**Solutions to Homework 4**

**Exercise 1*.** a) Since $X$ is continuous, it has a pdf $p_X$. We can write

$$1 - F_X(t) = \mathbb{P}(\{X > t\}) = \int_t^{+\infty} p_X(s)\, ds$$

and

$$\int_0^\infty (1 - F_X(t))\; dt = \int_0^\infty \left( \int_t^{+\infty} p_X(s)\, ds \right) dt = \int_0^\infty \left( \int_0^s p_X(s)\, dt \right) ds$$

where the second equality follows by exchanging integration order. Then

$$\int_0^\infty (1 - F_X(t))\; dt = \int_0^\infty [t p_X(s)]_0^s\, ds = \int_0^\infty s p_X(s) ds = \mathbb{E}(X)$$

For any continuous random variable we can write

$$\mathbb{E}(X) = \int_0^\infty s p_X(s) ds + \int_{-\infty}^0 s p_X(s) ds$$

and it remains to show that

$$\int_{-\infty}^0 s p_X(s) ds = -\int_{-\infty}^0 F_X(s)\, ds$$

Following asimilar sequence of steps

$$\begin{aligned}\int_{-\infty}^0 F_X(s)\, ds &= \int_{-\infty}^0 \left( \int_{-\infty}^s p_X(t)\, dt \right) ds = \int_{-\infty}^0 \left( \int_t^0 p_X(t)\, ds \right) dt \\ &= \int_{-\infty}^0 [s p_X(t)]_t^0\, dt = -\int_{-\infty}^0 t p_X(t) dt\end{aligned}$$

b)

$$\begin{aligned}\mathbb{E}(X) &= \int_0^\infty \frac{1}{2} \exp(-\lambda t) dt - \int_{-\infty}^0 \frac{1}{2} \exp(\lambda t) dt \\ &= \left[ -\frac{1}{\lambda} \frac{1}{2} \exp(-\lambda t) \right]_0^\infty - \left[ \frac{1}{\lambda} \frac{1}{2} \exp(\lambda t) \right]_{-\infty}^0 = \frac{1}{2\lambda} - \frac{1}{2\lambda} = 0\end{aligned}$$

c)

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k \geq 0} k \mathbb{P}(\{X = k\}) = \sum_{k \geq 0} k \left( \mathbb{P}(\{X > k-1\}) - \mathbb{P}(\{X > k\}) \right) \\ &= \sum_{k \geq 1} \left( k \mathbb{P}(\{X > k-1\}) - (k-1) \mathbb{P}(\{X > k-1\}) \right) \\ &= \sum_{k \geq 1} \mathbb{P}(\{X > k-1\}) = \sum_{k \geq 0} (1 - F_X(k))\end{aligned}$$

For $X \sim \text{Geom}(p)$ we have

$$\mathbb{E}(X) = \sum_{k\geq 0} (1 - F_X(k)) = \sum_{k\geq 0} (1-p)^{k+1} = \sum_{k\geq 0} (1-p)(1-p)^k = \frac{1-p}{p}$$

**Exercise 2.** a) We have

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a \, \lambda \, \exp(-\lambda x) \, dx < +\infty \quad \text{if and only if} \quad a > -1$$

b) Likewise:

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \int_0^{+\infty} x^{2a} \, \lambda \, \exp(-\lambda x) \, dx < +\infty \quad \text{if and only if} \quad a > -\frac{1}{2}$$

c) Therefore, c1) $\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$ is well defined and finite $\forall a > -\frac{1}{2}$; c2) $\text{Var}(Y)$ is well defined but takes the value $+\infty$ for $-\frac{1}{2} \geq a > -1$, and c3) $\text{Var}(Y)$ is ill-defined (indetermination of the type $\infty - \infty$) for $a \leq -1$.

d) The only integer values of $a$ for which $\mathbb{E}(Y)$ and $\text{Var}(Y)$ are well-defined are non-negative values. For $a = 0$, we have $Y = X^0 = 1$, so $\mathbb{E}(Y) = 1$ and $\text{Var}(Y) = 0$. For $a \geq 1$, we obtain by integration by parts:

$$\begin{aligned}\mathbb{E}(Y) = \mathbb{E}(X^a) &= \int_0^{+\infty} x^a \, \lambda \, \exp(-\lambda x) \, dx \\ &= \int_0^{+\infty} \frac{a}{\lambda} \, x^{a-1} \, \lambda \, \exp(-\lambda x) \, dx = \ldots = \frac{a!}{\lambda^a} \cdot 1\end{aligned}$$

so

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \frac{(2a)!}{\lambda^{2a}} \quad \text{and} \quad \text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{(2a)! - (a!)^2}{\lambda^{2a}}$$

**Exercise 3.** First note that as $X \sim -X$, it holds that $\mathbb{P}(\{X \geq 0\}) \geq \frac{1}{2}$ and $\mathbb{E}(X) = 0$.

a) $\text{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X\geq 0\}}) \geq 0$ as $X \, 1_{\{X\geq 0\}}$ is a non-negative random variable.

b) Using the suggested inequality, we find

$$\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X)} \, \sqrt{\text{Var}(Y)} = \sqrt{1} \, \sqrt{\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2} \leq \sqrt{\frac{1}{4}} = \frac{1}{2} = C$$

as $\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2 \leq \frac{1}{4}$ (which is maximized when $\mathbb{P}(\{X \geq 0\}) = \frac{1}{2}$).

c) The computation gives

$$\text{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X\geq 0\}}) = \int_0^{+\infty} x \, \frac{1}{\sqrt{2\pi}} \, \exp(-x^2/2) \, dx = \frac{1}{\sqrt{2\pi}} \, (-\exp(-x^2/2))\Big|_{x=0}^{x=+\infty} = \frac{1}{\sqrt{2\pi}}$$

(clearly satisfying the above two inequalities)

d) The answer to the first question is yes: take $X$ such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ (verifying $X \sim -X$, $\text{Var}(X) = 1$ and $\text{Cov}(X, Y) = \frac{1}{2}$).

e) The answer to the first question is no, but the one to the second is yes: consider $X_n$ such that $\mathbb{P}(\{X_n = n\}) = \mathbb{P}(\{X_n = -n\}) = \frac{1}{2n^2}$ and $\mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n^2}$. Then $X_n \sim -X_n$ and

$\text{Var}(X_n) = 1$ for every $n$, and $\text{Cov}(X_n, Y_n) = \mathbb{E}(X_n\, 1_{\{X_n \geq 0\}}) = n\, \frac{1}{2n^2} = \frac{1}{2n} \underset{n\to\infty}{\to} 0.$

**Exercise 4.** a) The computation of the characteristic function gives in this case:

$$\phi_X(t) = \sum_{k\geq 0} \frac{\lambda^k e^{-\lambda}}{k!}\, e^{itk} = \sum_{k\geq 0} \frac{(\lambda e^{it})^k e^{-\lambda}}{k!} = e^{\lambda e^{it}}\, e^{-\lambda} = e^{\lambda(e^{it}-1)}$$

b) The general expression for $\phi_X$ is given by

$$\phi_X(t) = \sum_{\ell\in\mathbb{Z}} \mathbb{P}(\{X = \ell\})\, e^{it\ell}$$

Plugging this expression into the proposed formula, we find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\, \phi_X(t)\, dt = \sum_{\ell\in\mathbb{Z}} \mathbb{P}(\{X = \ell\})\, \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it(\ell-k)}\, dt = \sum_{l\in\mathbb{Z}} \mathbb{P}(\{X = \ell\})\, \delta_{k\ell} = \mathbb{P}(\{X = k\})$$

where we have switched the sum and integral without too much checking and we have used the fact that for $k \neq \ell$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it(\ell-k)}\, dt = \frac{1}{2\pi}\, \frac{e^{ik(\ell-k)}}{i(\ell-k)}\Bigg|_{t=-\pi}^{t=\pi} = 0$$

c) Let us compute

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\, \cos(t)\, dt &= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{-itk}\, (e^{it} + e^{-it})\, dt \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{-it(k-1)} + e^{-it(k+1)})\, dt = \begin{cases} \frac{1}{2} & \text{if } k \in \{-1, +1\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by the same argument as above.

d) We know that $\phi_X(t) = \cos(t)$ is a characteristic function because $\phi_X(0) = \cos(0) = 1$, $\phi_X$ is continuous on $\mathbb{R}$, and also positive semi-definite. Indeed, using the trigonometric identity $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, we obtain

$$\begin{aligned} \sum_{j,k=1}^{n} c_j\, \overline{c_k}\, \phi_X(t_j - t_k) &= \sum_{j,k=1}^{n} c_j\, \overline{c_k}\, \cos(t_j - t_k) = \sum_{j,k=1}^{n} c_j\, \overline{c_k}\, (\cos(t_j)\, \cos(t_k) + \sin(t_j)\, \sin(t_k)) \\ &= \left|\sum_{j=1}^{n} c_j \cos(t_j)\right|^2 + \left|\sum_{j=1}^{n} c_j \sin(t_j)\right|^2 \geq 0 \end{aligned}$$

for every $n \geq 1$, $t_1, \ldots, t_n \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{C}$.