Solutions to Homework 3

Exercise 1. a) In this case,
\[ P(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1) \cdot \mu(B_2) = P(\{X_1 \in B_1\}) \cdot P(\{X_2 \in B_2\}) \]
The random variables \(X_1\) and \(X_2\) are therefore independent and identically distributed (i.i.d.).
b) In this case,
\[ P(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1 \cap B_2) \]
Note first that whenever \(B_1 \cap B_2 = \emptyset\), the above probability is zero, so it can never be the case that \(X_1, X_2\) take values simultaneously in disjoint sets \(B_1, B_2\). As this must hold for any disjoint sets \(B_1, B_2\), it holds in particular for non-intersecting intervals \([a_1, b_1[, [a_2, b_2[. This is to say that \(P(\{(X_1, X_2) \in R\}) = 0\) for any open rectangle \(R \subset \mathbb{R}^2\) not touching the diagonal \(\Delta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}\). From this, one deduces that \(P(\{(X_1, X_2) \in B\}) = 0\) for any open set \(B\) not touching the diagonal, which further implies that \(P(\{(X_1, X_2) \in \Delta\}) = 1\), i.e., that \(P(\{(X_1 = X_2\}) = 1\).

NB: Please note that in both cases, the two random variables \(X_1, X_2\) have the same distribution, but in one case, they are independent, while in the other, they are the same random variable.

Exercise 2. By the formula seen in class, we have:
\[
p_{X_1+X_2}(t) = \int_{\mathbb{R}} dx_1 p_{X_1}(x_1) p_{X_2}(t-x_1) = \int_{\mathbb{R}} dx_1 \frac{1}{\sqrt{2\pi}} \exp(-x_1^2/2) \frac{1}{\sqrt{2\pi}} \exp(-(t-x_1)^2/2)
\]
\[
= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \frac{1}{\sqrt{2\pi}} \exp(tx_1-x_1^2)
\]
\[
= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \frac{1}{\sqrt{2\pi}} \exp(-(x_1-t/2)^2) \exp(t^2/4)
\]
\[
= \frac{1}{\sqrt{4\pi}} \exp(-t^2/4) \int_{\mathbb{R}} dx_1 \frac{1}{\sqrt{\pi}} \exp(-(x_1-t/2)^2)
\]
The integral on the right-hand side is equal to 1, as the integrand is the pdf of a \(\mathcal{N}(t/2, 1/2)\) random variable, so we remain with
\[
p_{X_1+X_2}(t) = \frac{1}{\sqrt{4\pi}} \exp(-t^2/4), \quad t \in \mathbb{R}
\]
which shows that \(X_1 + X_2\) is a \(\mathcal{N}(0, 2)\) random variable.

Exercise 3. a) Yes. Because \(Y \sim \mathcal{N}(0, 1)\) and \(Z\) is independent of \(Y\), \(ZY \sim \mathcal{N}(0, 1)\); then, the sum of two independent Gaussians random variables is also Gaussian.
b) No. For example, \(P(\{X + ZY \geq 0\}) P(\{Y \geq 0\}) = 1/4\) by symmetry, but
\[
P(\{X + ZY \geq 0, Y \geq 0\}) = \frac{1}{2} P(\{X + Y \geq 0, Y \geq 0\}) + \frac{1}{2} P(\{X - Y \geq 0, Y \geq 0\})
\]
\[
= P(\{X \geq Y, Y \geq 0\}) + \frac{1}{2} P(\{|X| \leq Y, Y \geq 0\}) > \frac{1}{4}
\]
Exercise 4.  

a) Yes, $Y_2$ and $Y_3$ are independent. By inspection, 

$$
P(Y_2 = i, Y_3 = j) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} 
= P(Y_2 = i)P(Y_3 = j)
$$

for all $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$.

b) Let $A_0 = \{2, 4, 6\}$ and $A_1 = A_0^c = \{1, 3, 5\}$. Then, the $\sigma$-field generated by $Y_2$ is the $\sigma$-field generated by the atoms $A_0, A_1$. That is, $\sigma(Y_2) = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$.

Likewise, let $B_0 = \{3, 6\}, B_1 = \{1, 4\}$, and $B_2 = \{2, 5\}$. The $\sigma$-field generated by $Y_3$ is the $\sigma$-field generated by the atoms $B_0, B_1, B_2$.

That is, $\sigma(Y_3) = \{\emptyset, \{3, 6\}, \{2, 5\}, \{1, 4\}, \{2, 3, 5, 6\}, \{1, 2, 4, 5\}, \{1, 3, 4, 6\}, \Omega\}$.

c) Yes to both. The random variables $Y_2, Y_3, Y_5$ are pairwise independent and jointly independent. Thus, it is sufficient to show that they are jointly independent. This can be done by considering the $\sigma$-fields generated by each random variable and checking the definition of independence. Alternatively, we can show from definition 3.7 in lecture notes that three random variables are jointly independent if and only if the pmf factorizes (in the same way as we did this with pairwise independence). Thus, in this case 

$$
P(Y_2 = i, Y_3 = j, Y_5 = k) = \frac{1}{30} = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{5} 
= P(Y_2 = i)P(Y_3 = j)P(Y_5 = k)
$$

for all $i \in \{0, 1\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1, 2, 3, 4\}$. The first equation follows from the fact that a unique number in $\{1, \ldots, 30\}$ has remainders $(i, j, k)$ when divided by $2, 3,$ and $5$, respectively. This can be seen by inspection, or more generally, by the Chinese Remainder Theorem.