Solutions to Homework 3

Exercise 1. a) In this case,

$$\mathbb{P}^{(1)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1) \cdot \mu(B_2) = \mathbb{P}^{(1)}(\{X_1 \in B_1\}) \cdot \mathbb{P}^{(1)}(\{X_2 \in B_2\})$$

The random variables X_1 and X_2 are therefore independent and identically distributed (i.i.d.).

b) In this case,

$$\mathbb{P}^{(2)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1 \cap B_2)$$

Note first that whenever $B_1 \cap B_2 = \emptyset$, the above probability is zero, so it can never be the case that X_1, X_2 take values simultaneously in disjoint sets B_1, B_2. As this must hold for *any* disjoint sets B_1, B_2, it holds in particular for non-intersecting intervals $]a_1, b_1[$, $]a_2, b_2[$. This is to say that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in R\}) = 0$ for any open rectangle $R \subset \mathbb{R}^2$ not touching the diagonal $\Delta = \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 = x_2\}$. From this, one deduces that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in B\}) = 0$ for any open set B not touching the diagonal, which further implies that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in \Delta\}) = 1$, i.e., that $\mathbb{P}^{(2)}(\{X_1 = X_2\}) = 1$.

NB: Please note that in both cases, the two random variables X_1, X_2 have the same *distribution*, but in one case, they are independent, while in the other, they are the same random variable.

Exercise 2. By the formula seen in class, we have:

$$\begin{aligned}
p_{X_1+X_2}(t) &= \int_{\mathbb{R}} dx_1 \, p_{X_1}(x_1) \, p_{X_2}(t - x_1) = \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(-x_1^2/2) \, \frac{1}{\sqrt{2\pi}} \exp(-(t - x_1)^2/2) \\
&= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(tx_1 - x_1^2) \\
&= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{2\pi}} \exp(-(x_1 - t/2)^2) \exp(t^2/4) \\
&= \frac{1}{\sqrt{4\pi}} \exp(-t^2/4) \int_{\mathbb{R}} dx_1 \, \frac{1}{\sqrt{\pi}} \exp(-(x_1 - t/2)^2)
\end{aligned}$$

The integral on the right-hand side is equal to 1, as the integrand is the pdf of a $\mathcal{N}(t/2, 1/2)$ random variable, so we remain with

$$p_{X_1+X_2}(t) = \frac{1}{\sqrt{4\pi}} \exp(-t^2/4), \quad t \in \mathbb{R}$$

which shows that $X_1 + X_2$ is a $\mathcal{N}(0, 2)$ random variable.

Exercise 3. a) Yes. Because $Y \; \mathcal{N}(0, 1)$ and Z is independent of Y, $ZY \sim \mathcal{N}(0, 1)$; then, the sum of two independent Gaussians random variables is also Gaussian.

b) No. For example, $\mathbb{P}(\{X + ZY \geq 0\}) \, \mathbb{P}(\{Y \geq 0\}) = 1/4$ by symmetry, but

$$\begin{aligned}
\mathbb{P}(\{X + ZY \geq 0, Y \geq 0\}) &= \frac{1}{2} \mathbb{P}(\{X + Y \geq 0, Y \geq 0\}) + \frac{1}{2} \mathbb{P}(\{X - Y \geq 0, Y \geq 0\}) \\
&= \mathbb{P}(\{X \geq Y, Y \geq 0\}) + \frac{1}{2} \mathbb{P}(\{|X| \leq Y, Y \geq 0\}) > \frac{1}{4}
\end{aligned}$$

Exercise 4. a) Yes, Y_2 and Y_3 are independent. By inspection,

$$\begin{aligned}\mathbb{P}(Y_2 = i, Y_3 = j) &= \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} \\ &= \mathbb{P}(Y_2 = i)\mathbb{P}(Y_3 = j)\end{aligned}$$

for all $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$.

b) Let $A_0 = \{2, 4, 6\}$ and $A_1 = A_0^c = \{1, 3, 5\}$. Then, the σ-field generated by Y_2 is the σ-field generated by the atoms A_0, A_1. That is, $\sigma(Y_2) = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$.

Likewise, let $B_0 = \{3, 6\}$, $B_1 = \{1, 4\}$, and $B_2 = \{2, 5\}$. The σ-field generated by Y_3 is the σ-field generated by the atoms B_0, B_1, B_2.
That is, $\sigma(Y_3) = \{\emptyset, \{3, 6\}, \{2, 5\}, \{1, 4\}, \{2, 3, 5, 6\}, \{1, 2, 4, 5\}, \{1, 3, 4, 6\}, \Omega\}$.

c) Yes to both. The random variables Y_2, Y_3, Y_5 are pairwise independent and jointly independent. Thus, it is sufficient to show that they are jointly independent. This can be done by considering the σ-fields generated by each random variable and checking the definition of independence. Alternatively, we can show from definition 3.7 in lecture notes that three random variables are jointly independent if and only if the pmf factorizes (in the same way as we did this with pairwise independence). Thus, in this case

$$\begin{aligned}\mathbb{P}(Y_2 = i, Y_3 = j, Y_5 = k) &= \frac{1}{30} = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{5} \\ &= \mathbb{P}(Y_2 = i)\mathbb{P}(Y_3 = j)\mathbb{P}(Y_5 = k)\end{aligned}$$

for all $i \in \{0, 1\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1, 2, 3, 4\}$. The first equation follows from the fact that a unique number in $\{1, \ldots, 30\}$ has remainders (i, j, k) when divided by 2, 3, and 5, respectively. This can be seen by inspection, or more generally, by the *Chinese Remainder Theorem*.