Solution Set 3

Problem 1: Events and independence

Let $n \geq 1$, $\Omega = \{1, 2, \ldots, n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ defined by $\mathbb{P}(\{\omega\}) = \frac{1}{n}$ on the singletons and extended by additivity to all subsets of Ω.

a) Consider first $n = 4$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{but} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

b) Consider now $n = 6$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \quad \text{but} \quad \exists j \neq k \text{ such that } \mathbb{P}(A_j \cap A_k) \neq \mathbb{P}(A_j) \cdot \mathbb{P}(A_k)$$

c) Consider finally a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and three events $A_1, A_2, A_3 \in \mathcal{F}$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{and} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

Show that A_1, A_2, A_3 are independent according to the definition given in the course.

Solution a) Here are 3 possible subsets A_1, A_2, A_3 of $\Omega = \{1, 2, 3, 4\}$: $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$ and $A_3 = \{1, 4\}$. We check that

$$\mathbb{P}(A_j) = \frac{1}{2} \quad \forall j \quad \text{and} \quad \mathbb{P}(A_j \cap A_k) = \frac{1}{4} = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k$$

but

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

b) Here are 3 possible subsets A_1, A_2, A_3 of $\Omega = \{1, 2, 3, 4, 5, 6\}$: $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$ and $A_3 = \{1, 3, 4, 6\}$. We check that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

but

$$\mathbb{P}(A_1 \cap A_2) = \frac{1}{6} \neq \frac{1}{4} = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$$

c) Using the assumptions made, we check successively (the roles of A_1, A_2, A_3 being permutable):

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2 \cap A_3^c) &= \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \\
&= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot (1 - \mathbb{P}(A_3)) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3^c) \\
\mathbb{P}(A_1 \cap A_2^c \cap A_3^c) &= \mathbb{P}(A_1 \cap A_3^c) - \mathbb{P}(A_1 \cap A_2 \cap A_3^c) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_3^c) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3^c) \\
&= \mathbb{P}(A_1) \cdot (1 - \mathbb{P}(A_2)) \cdot \mathbb{P}(A_3^c) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) \\
\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) &= \mathbb{P}(A_2^c \cap A_3^c) - \mathbb{P}(A_1 \cap A_2^c \cap A_3^c) = \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) - \mathbb{P}(A_1) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) \\
&= (1 - \mathbb{P}(A_1)) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) = \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c)
\end{aligned}$$

Problem 2: Sigma field and independence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{(\omega_1, \omega_2) \colon \omega_1, \omega_2 \in \{1, 2, \ldots, n\}\}$ for some $n \geq 1$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1, \omega_2) = \frac{1}{n^2}$ for all $(\omega_1, \omega_2) \in \Omega$.

a) Let $X_1 = \omega_1 + \omega_2$. Describe $\sigma(\{X_1\})$, the σ-field generated by X_1. How many atoms does it have? What are they?

b) Let $X_2 = \omega_1 - \omega_2$. Are X_1 and X_2 independent? Why or why not?

c) Let $X = \omega_1$, $Z = 1_{\{\omega_1 = \omega_2\}}$, and $Y = 1_{\{\omega_1 + \omega_2 = n+1\}}$. Are X, Y, Z pairwise independent? Why or why not?

Solution a) The atoms of $\sigma(\{X_1\})$ have the form $S_j = \{w_1, w_2 \colon w_1 + w_2 = j\}$ for $j = 2, \ldots, 2n$. Thus, it has $2n-1$ atoms, and consists of 2^{2n-1} subsets generated by every possible union of these atoms.

b) No, X_1 and X_2 are not independent unless $n = 1$. For example,

$$\mathbb{P}\left(X_1 = 2, X_2 = 0\right) = \mathbb{P}\left(\{(\omega_1, \omega_2) = (1,1)\}\right) = \frac{1}{n^2}.$$

On the other hand

$$\mathbb{P}\left(X_1 = 2\right) \mathbb{P}\left(X_2 = 0\right) = \frac{1}{n^2} \cdot \frac{1}{n}.$$

c) It is always true that 1) $X \perp\!\!\!\perp Z$ and $X \perp\!\!\!\perp Y$. 2) For n even Z and Y are not independent. 3) For n odd, we also have that $Z \perp\!\!\!\perp Y$.

1) $X \perp\!\!\!\perp Z$:

$$\mathbb{P}\left(X = j, Z = 1\right) = \mathbb{P}\left(\{(\omega_1, \omega_2) = (j, j)\}\right) = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}\left(X = j\right) \mathbb{P}\left(Z = 1\right)$$

and

$$\mathbb{P}\left(X = j, Z = 0\right) = \mathbb{P}\left(\{(\omega_1, \omega_2) = (j, k) \colon k \neq j\}\right) = \frac{n-1}{n^2} = \frac{1}{n} \cdot \frac{n-1}{n} = \mathbb{P}\left(X = j\right) \mathbb{P}\left(Z = 0\right)$$

Note that $X \perp\!\!\!\perp Y$ follows by a completely symmetric argument.

2) For n even Z and Y are not independent. We have

$$\mathbb{P}\left(Z = 1, Y = 1\right) = 0 \neq \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}\left(Z = 1\right) \mathbb{P}\left(Y = 1\right)$$

3) For n odd, we also have that $Z \perp\!\!\!\perp Y$:

$$\mathbb{P}\left(Z = 1, Y = 1\right) = \mathbb{P}\left(\left\{(\omega_1, \omega_2) = \left(\frac{n+1}{2}, \frac{n+1}{2}\right)\right\}\right) = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}\left(Z = 1\right) \mathbb{P}\left(Y = 1\right)$$

also

$$\mathbb{P}\left(Z = 0, Y = 0\right) = \frac{n^2 - 2n + 1}{n^2} = \frac{n-1}{n} \cdot \frac{n-1}{n} = \mathbb{P}\left(Z = 0\right) \mathbb{P}\left(Y = 0\right)$$

and

$$\mathbb{P}\left(Z = 1, Y = 0\right) = \mathbb{P}\left(\left\{(\omega_1, \omega_2) = (j, j), j \neq \frac{n+1}{2}\right\}\right) = \frac{n-1}{n^2} = \frac{1}{n} \cdot \frac{n-1}{n} = \mathbb{P}\left(Z = 1\right) \mathbb{P}\left(Y = 0\right).$$

Finally, the case with $\mathbb{P}\left(Z = 0, Y = 1\right)$ follows by symmetry.

Problem 3: Extending probability measures

Let $\Omega = \mathbb{R}^2$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Let also $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally μ be a probability distribution on $\mathbb{R}$. We consider below two different probability measures defined on $(\Omega, \mathcal{F})$, defined on the "rectangles" $B_1 \times B_2$ (Caratheodory's extension theorem then guarantees that these probability measures can be extended uniquely to $\mathcal{B}(\mathbb{R}^2)$).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables X_1 and X_2.

Solution a) In this case,

$$\mathbb{P}^{(1)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1) \cdot \mu(B_2) = \mathbb{P}^{(1)}(\{X_1 \in B_1\}) \cdot \mathbb{P}^{(1)}(\{X_2 \in B_2\})$$

The random variables X_1 and X_2 are therefore independent and identically distributed (i.i.d.).

b) In this case,

$$\mathbb{P}^{(2)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1 \cap B_2)$$

Note first that whenever $B_1 \cap B_2 = \emptyset$, the above probability is zero, so it can never be the case that X_1, X_2 take values simultaneously in disjoint sets B_1, B_2. As this must hold for *any* disjoint sets B_1, B_2, it holds in particular for non-intersecting intervals $]a_1, b_1[$, $]a_2, b_2[$. This is to say that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in R\}) = 0$ for any open rectangle $R \subset \mathbb{R}^2$ not touching the diagonal $\Delta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. From this, one deduces that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in B\}) = 0$ for any open set B not touching the diagonal, which further implies that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in \Delta\}) = 1$, i.e., that $\mathbb{P}^{(2)}(\{X_1 = X_2\}) = 1$.

NB: Please note that in both cases, the two random variables X_1, X_2 have the same *distribution*, but in one case, they are independent, while in the other, they are the same random variable.

Problem 4: Exponential random variables

Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{E}(1)$ random variables (i.e., they are independent and identically distributed, all with the exponential distribution of parameter $\lambda = 1$).

a) Compute the cdf of $Y_n = \min\{X_1, \ldots, X_n\}$.

b) How do $\mathbb{P}(\{Y_n \leq t\})$ and $\mathbb{P}(\{X_1 \leq t\})$ compare when n is large and t is such that $t \ll \frac{1}{n}$?

c) Compute the cdf of $Z_n = \max\{X_1, \ldots, X_n\}$.

d) How do $\mathbb{P}(\{Z_n \geq t\})$ and $\mathbb{P}(\{X_1 \geq t\})$ compare when n is large and t is such that $t \gg \log(n)$?

Solution a) We have

$$\begin{aligned}\mathbb{P}(\{Y_n \leq t\}) &= 1 - \mathbb{P}(\{Y_n > t\}) = 1 - \mathbb{P}(\{\min\{X_1, \ldots, X_n\} > t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j > t\}) \\ &= 1 - \textstyle\prod_{j=1}^n \mathbb{P}(\{X_j > t\}) = 1 - \mathbb{P}(\{X_1 > t\})^n\end{aligned}$$

where the last two equalities follow from the assumption that the X's are i.i.d. Therefore,

$$\mathbb{P}(\{Y_n \leq t\}) = 1 - (\exp(-t))^n = 1 - \exp(-nt)$$

b) Under the assumptions made, n is large and t is such that $nt \ll 1$, so using Taylor's expansion $\exp(-x) \simeq 1 - x$, we obtain

$$\mathbb{P}(\{Y_n \le t\}) \simeq 1 - (1 - nt) = nt \quad \text{while} \quad \mathbb{P}(\{X_1 \le t\}) = 1 - \exp(-t) \simeq t$$

and therefore $\mathbb{P}(\{Y_n \le t\}) \simeq n\,\mathbb{P}(\{X_1 \le t\})$.

c) We have similarly

$$\begin{aligned} \mathbb{P}(\{Z_n \ge t\}) &= 1 - \mathbb{P}(\{Z_n < t\}) = 1 - \mathbb{P}(\{\max\{X_1, \ldots, X_n\} < t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j < t\}) \\ &= 1 - \prod_{j=1}^n \mathbb{P}(\{X_j < t\}) = 1 - \mathbb{P}(\{X_1 < t\})^n = 1 - (1 - \exp(-t))^n \end{aligned}$$

d) Under the assumptions made, n is large and t is such that $n\exp(-t) \ll 1$, so using again the same Taylor expansion as above, we obtain

$$\mathbb{P}(\{Z_n \ge t\}) \simeq 1 - (1 - n\,\exp(-t)) = n\,\exp(-t) \quad \text{while} \quad \mathbb{P}(\{X_1 \ge t\}) = \exp(-t)$$

and therefore $\mathbb{P}(\{Z_n \ge t\}) \simeq n\,\mathbb{P}(\{X_1 \ge t\})$.