Advanced Probability & Applications
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Problem Set 3

Problem 1: Events and independence

Let $n \geq 1$, $\Omega = \{1, 2, \ldots, n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ defined by $\mathbb{P}(\{\omega\}) = \frac{1}{n}$ on the singletons and extended by additivity to all subsets of Ω.

a) Consider first $n = 4$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{but} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

b) Consider now $n = 6$. Find three subsets $A_1, A_2, A_3 \subset \Omega$ such that

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \quad \text{but} \quad \exists j \neq k \text{ such that } \mathbb{P}(A_j \cap A_k) \neq \mathbb{P}(A_j) \cdot \mathbb{P}(A_k)$$

c) Consider finally a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and three events $A_1, A_2, A_3 \in \mathcal{F}$ such that

$$\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{and} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

Show that A_1, A_2, A_3 are independent according to the definition given in the course.

Problem 2: Sigma field and independence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{(\omega_1, \omega_2) \colon \omega_1, \omega_2 \in \{1, 2, \ldots, n\}\}$ for some $n \geq 1$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1, \omega_2) = \frac{1}{n^2}$ for all $(\omega_1, \omega_2) \in \Omega$.

a) Let $X_1 = \omega_1 + \omega_2$. Describe $\sigma(\{X_1\})$, the σ-field generated by X_1. How many atoms does it have? What are they?

b) Let $X_2 = \omega_1 - \omega_2$. Are X_1 and X_2 independent? Why or why not?

c) Let $X = \omega_1$, $Z = 1_{\{\omega_1 = \omega_2\}}$, and $Y = 1_{\{\omega_1 + \omega_2 = n+1\}}$. Are X, Y, Z pairwise independent? Why or why not?

Problem 3: Extending probability measures

Let $\Omega = \mathbb{R}^2$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Let also $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally μ be a probability distribution on $\mathbb{R}$. We consider below two different probability measures defined on $(\Omega, \mathcal{F})$, defined on the "rectangles" $B_1 \times B_2$ (Caratheodory's extension theorem then guarantees that these probability measures can be extended uniquely to $\mathcal{B}(\mathbb{R}^2)$).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables X_1 and X_2.

Problem 4: Exponential random variables

Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{E}(1)$ random variables (i.e., they are independent and identically distributed, all with the exponential distribution of parameter $\lambda = 1$).

a) Compute the cdf of $Y_n = \min\{X_1, \dots, X_n\}$.

b) How do $\mathbb{P}(\{Y_n \leq t\})$ and $\mathbb{P}(\{X_1 \leq t\})$ compare when n is large and t is such that $t \ll \frac{1}{n}$?

c) Compute the cdf of $Z_n = \max\{X_1, \dots, X_n\}$.

d) How do $\mathbb{P}(\{Z_n \geq t\})$ and $\mathbb{P}(\{X_1 \geq t\})$ compare when n is large and t is such that $t \gg \log(n)$?