Homework 3

Exercise 1. Let $\Omega = \mathbb{R}^2$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Let also $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally μ be a probability distribution on $\mathbb{R}$. We consider below two different probability measures defined on $(\Omega, \mathcal{F})$, defined on the "rectangles" $B_1 \times B_2$ (Caratheodory's extension theorem then guarantees that these probability measures can be extended uniquely to $\mathcal{B}(\mathbb{R}^2)$).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables X_1 and X_2.

Exercise 2. Let X_1, X_2 be two independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Compute the pdf of $X_1 + X_2$ (using convolution).

Exercise 3. Let X, Y be two i.i.d. $\mathcal{N}(0,1)$ random variables, and Z be independent of X, Y and such that $\mathbb{P}\{Z = +1\}) = \mathbb{P}\{Z = -1\}) = 1/2$.

a) Is it true $X + Z\,Y$ is a Gaussian random variable ?

b) Is it true that $X + Z\,Y$ and Y are independent random variables ?

Exercise 4. Let $\Omega = \{1, \ldots n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\{\omega\}) = \frac{1}{n}$ for all $\omega \in \Omega$. Let $X(\omega) = \omega$.

a) Let $n = 6$, and define

$$Y_2 = X \mod 2, \quad \text{and} \quad Y_3 = X \mod 3.$$

Are Y_2 and Y_3 independent?

b) Describe $\sigma(\{Y_2\})$ and $\sigma(\{Y_3\})$.

c) Let $n = 30$, and define

$$Y_2 = X \mod 2, \quad Y_3 = X \mod 3, \quad \text{and} \quad Y_5 = X \mod 5.$$

Are Y_i and Y_j pairwise independent for $i \neq j$, $i, j \in \{2, 3, 5\}$? Are Y_2, Y_3, Y_5 (jointly) independent?

Exercise 5.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{(\omega_1, \omega_2) \colon \omega_1, \omega_2 \in \{1, 2, \ldots, n\}\}$ for some $n \geq 1$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1, \omega_2) = \frac{1}{n^2}$ for all $(\omega_1, \omega_2) \in \Omega$.

a) Let $X_1 = \omega_1 + \omega_2$. Describe $\sigma(\{X_1\})$, the σ-field generated by X_1. How many atoms does it have? What are they?

b) Let $X_2 = \omega_1 - \omega_2$. Are X_1 and X_2 independent? Why or why not?

c) Let $X = \omega_1$, $Z = 1_{\{\omega_1 = \omega_2\}}$, and $Y = 1_{\{\omega_1 + \omega_2 = n+1\}}$. Are X, Y, Z pairwise independent? Why or why not?

Exercise 6. Let X, Y be two discrete random variables, each with values in $\{0,1\}$.

a) Show that $X \perp\!\!\!\perp Y$ if $\mathbb{P}(\{Y=1\}|\{X=0\}) = \mathbb{P}(\{Y=1\}|\{X=1\})$.

Let moreover $Z = X \oplus Y = \begin{cases} 1, & \text{if } X=1, Y=0 \text{ or } X=0, Y=1, \\ 0, & \text{otherwise.} \end{cases}$

b) Show that $X \perp\!\!\!\perp Z$ if $\mathbb{P}(\{Y=1\}|\{X=0\}) = \mathbb{P}(\{Y=0\}|\{X=1\})$.

c) Which assumption guarantees that both $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Z$?

d) Assume that none of the 3 random variables X, Y, Z is constant (i.e., takes a single value with probability 1). Can it be that the collection of the three random variables (X, Y, Z) is independent? Justify your answer.