Solutions to Homework 2

**Exercise 1.** a) 1. true, 2. false, 3. false, 4. true   b) 5. false, 6. true, 7. false, 8. true.

**Exercise 2.** a) We have
\[
P(\{Y_n \leq t\}) = 1 - P(\{Y_n > t\}) = 1 - P(\{\min\{X_1, \ldots, X_n\} > t\}) = 1 - P(\bigcap_{j=1}^n \{X_j > t\})
\]
\[= 1 - \prod_{j=1}^n P(\{X_j > t\}) = 1 - P(\{X_1 > t\})^n
\]
where the last two equalities follow from the assumption that the $X$'s are i.i.d. Therefore,
\[
P(\{Y_n \leq t\}) = 1 - (\exp(-t))^n = 1 - \exp(-nt)
\]
b) Under the assumptions made, $n$ is large and $t$ is such that $nt \ll 1$, so using Taylor’s expansion $\exp(-x) \simeq 1 - x$, we obtain
\[
P(\{Y_n \leq t\}) \simeq 1 - (1 - nt) = nt \quad \text{while} \quad P(\{X_1 \leq t\}) = 1 - \exp(-t) \simeq t
\]
and therefore $P(\{Y_n \leq t\}) \simeq nP(\{X_1 \leq t\})$.

c) We have similarly
\[
P(\{Z_n \geq t\}) = 1 - P(\{Z_n < t\}) = 1 - P(\{\max\{X_1, \ldots, X_n\} < t\}) = 1 - P(\bigcap_{j=1}^n \{X_j < t\})
\]
\[= 1 - \prod_{j=1}^n P(\{X_j < t\}) = 1 - P(\{X_1 < t\})^n = 1 - (1 - \exp(-t))^n
\]
d) Under the assumptions made, $n$ is large and $t$ is such that $n\exp(-t) \ll 1$, so using again the same Taylor expansion as above, we obtain
\[
P(\{Z_n \geq t\}) \simeq 1 - (1 - n\exp(-t)) = n\exp(-t) \quad \text{while} \quad P(\{X_1 \geq t\}) = \exp(-t)
\]
and therefore $P(\{Z_n \geq t\}) \simeq nP(\{X_1 \geq t\})$.

**Exercise 3.**

a) No. Take for example $\Omega = \{1, 2, 3\}$, $X(\omega) = \omega$ and $Y(\omega) = 2$ for every $\omega \in \Omega$. Then $\mathcal{G} = \sigma(X) \cap \sigma(Y) = \sigma(Y) = \{\emptyset, \Omega\}$, but $\{X \leq Y\} = \{\omega \in \Omega : X(\omega) \leq Y(\omega)\} = \{1, 2\} \notin \mathcal{G}$.

b) No. Take for example $\Omega = \{1, 2\}^2$, $X(\omega) = \omega_1$ and $Y(\omega) = -\omega_2$. Then $\{X + Y = 0\} = \{(1, 1), (2, 2)\}$, and so $\sigma(X + Y) = \sigma(\{(1, 1), (2, 2)\}, \{(1, 2), (2, 1)\}) \neq \sigma(X, Y) = \mathcal{P}(\Omega)$ (in addition, note that the fact that $X$ and $Y$ are independent does not play a role here).

c) No. Take for example $X \sim \mathcal{N}(0, 1)$, whose pdf $p_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ is continuous. Then $Y = X^2$ has pdf
\[
p_Y(y) = \begin{cases} 
\frac{1}{\sqrt{2\pi y}} \exp(-y/2) & \text{if } y \geq 0 \\
0 & \text{if } y < 0
\end{cases}
\]
which is discontinuous in $y = 0$. 

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d) Yes. Actually, the map \( t \mapsto t^3 + 3t^2 + 3t + 1 = (t + 1)^3 \) is non-decreasing and going from \(-\infty\) to \(+\infty\), thus the properties of the cdf \( F \) are preserved for \( G \).

**Exercise 4.** a) Here are 3 possible subsets \( A_1, A_2, A_3 \) of \( \Omega = \{1, 2, 3, 4\} \): \( A_1 = \{1, 2\}, A_2 = \{1, 3\} \) and \( A_3 = \{1, 4\} \). We check that

\[
P(A_j) = \frac{1}{2} \quad \forall j \quad \text{and} \quad P(A_j \cap A_k) = \frac{1}{4} = P(A_j) \cdot P(A_k) \quad \forall j \neq k
\]

but

\[
P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \frac{1}{8} = P(A_1) \cdot P(A_2) \cdot P(A_3)
\]

b) Here are 3 possible subsets \( A_1, A_2, A_3 \) of \( \Omega = \{1, 2, 3, 4, 5, 6\} \): \( A_1 = \{1, 2, 3\}, A_2 = \{3, 4, 5\} \) and \( A_3 = \{1, 3, 4, 6\} \). We check that

\[
P(A_1 \cap A_2 \cap A_3) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = P(A_1) \cdot P(A_2) \cdot P(A_3)
\]

but

\[
P(A_1 \cap A_2) = \frac{1}{6} \neq \frac{1}{4} = P(A_1) \cdot P(A_2)
\]

c) Using the assumptions made, we check successively (the roles of \( A_1, A_2, A_3 \) being permutable):

\[
P(A_1 \cap A_2 \cap A_3^c) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) - P(A_1) \cdot P(A_2) \cdot P(A_3)
\]

\[
= P(A_1) \cdot P(A_2) \cdot (1 - P(A_3)) = P(A_1) \cdot P(A_2) \cdot P(A_3)
\]

\[
P(A_1 \cap A_2^c \cap A_3^c) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3^c) = P(A_1) \cdot P(A_2) - P(A_1) \cdot P(A_2) \cdot P(A_3^c)
\]

\[
= P(A_1) \cdot (1 - P(A_2)) \cdot P(A_3^c) = P(A_1) \cdot P(A_2^c) \cdot P(A_3^c)
\]

\[
P(A_1^c \cap A_2^c \cap A_3^c) = P(A_1^c) \cdot P(A_2^c) - P(A_1 \cap A_2 \cap A_3^c) = P(A_1^c) \cdot P(A_2^c) - P(A_1) \cdot P(A_2^c) \cdot P(A_3^c)
\]

\[
= (1 - P(A_1)) \cdot P(A_2^c) \cdot P(A_3^c) = P(A_1) \cdot P(A_2^c) \cdot P(A_3^c)
\]

**Exercise 5.** a) No. Even though it is easily shown that \( Y \) and \( Z \) are uncorrelated random variables (i.e., that their covariance is zero), they are not independent. Here is a counter-example: \( P(\{Y = +2\}) = P(\{Z = +2\}) = 1/4 \), but \( P(\{Y = +2, Z = +2\}) = 0 \). So we have found two Borel sets \( B_1 = \{+2\} \) and \( B_2 = \{+2\} \) such that

\[
P(\{Y \in B_1, Z \in B_2\}) \neq P(\{Y \in B_1\}) \cdot P(\{Z \in B_2\})
\]

b) Yes. In this case again, one checks easily that \( Y \) and \( Z \) are uncorrelated. Let us now compute their joint pdf: the joint pdf of \( X_1 \) and \( X_2 \) is given by

\[
p_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \exp\left( -\frac{x_1^2 + x_2^2}{2} \right)
\]

Making now the change of variables \( y = x_1 + x_2 \), \( z = x_1 - x_2 \), or equivalently \( x_1 = \frac{y+z}{2}, x_2 = \frac{y-z}{2} \), we obtain

\[
x_1^2 + x_2^2 = \left( \frac{y+z}{2} \right)^2 + \left( \frac{y-z}{2} \right)^2 = \frac{y^2 + z^2}{2}
\]
and the Jacobian of this linear transformation is given by

$$ J(y, z) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{pmatrix} = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2} $$

so that

$$ p_{Y, Z}(y, z) = p_{X_1, X_2}(x_1(y, z), x_2(y, z)) \cdot |J(y, z)| = \frac{1}{4\pi} \exp \left( -\frac{y^2 + z^2}{4} \right) $$

from which we deduce that $Y$ and $Z$ are independent $\mathcal{N}(0, 2)$ random variables.