Solutions to Homework 1

Exercise 1.1* a) We can be more general and show that a countable union of countable sets is countable. That is, let $E_n, n = 1, 2, 3, \ldots$, be a sequence of countable sets and put

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Every set E_n can be arranged in a sequence $\{x_{n,k}\}, k = 1, 2, 3, \ldots$. Moreover, we can construct an infinite array

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

The array contains elements of A which can be numbered as $x_{11}; x_{21}, x_{31}, x_{22}, x_{13}; \ldots$, and so on. We can likewise show that a union of two countable sets is countable.

Observe that $\mathbb{Q}$ is a special case of a countable union of countable sets since for every positive $r \in \mathbb{Q}$ we can write $r = \frac{m}{n}$ for some positive integers m and n. Thus, we can construct a similar array with the m indexing the rows, and n indexing the columns (and skipping over any duplicates). By this argument, the positive rational numbers are at most countable. They are not finite since they include positive integers as a subset. Likewise, non-positive rational numbers are also countable. Finally, $\mathbb{Q}$ is countable since it is just a union of two countable sets.

b) Let $E \in A$ and suppose that E is infinite. Since A is countable, we can arrange its elements in a sequence $\{x_n\}$ of distinct elements. Construct a new sequence $\{n_k\}$ by letting n_1 be the smallest integer such that $x_{n_1} \in E$, n_2 the next smallest, and so on. Putting $f(k) = x_{n_k}, k = 1, 2, \ldots$, we obtain a 1-1 correspondence between E an positive integers.

c) The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$. Suppose A is countable, and let $s_1, s_2, s_3, \ldots$ be the sequence of all elements of A. We construct a new sequence s as follows. If the nth digit of s_n is 1, we let the nth digits of s be 0, and vice versa. Thus s is not equal to any element in the sequence, and $s \notin A$. This is a contradiction and so A is not countable.

If we represent real numbers on the interval $[0, 1]$ with their binary expansion, we get exactly the set A. Thus, the interval $[0, 1]$ is not countable. The interval $[0, 1]$ is an infinite subset of $\mathbb{R}$ and therefore (by part b), $\mathbb{R}$ is not countable.

d) Assume that the irrational numbers are countable. Then, $\mathbb{R}$ can be represented as a union of two countable sets and is countable (by part a). We have already shown that $\mathbb{R}$ is not countable in part c) so this is a contradiction. Therefore, irrational numbers are not countable.

Exercise 1.2 One way to solve this problem is to find a set that is larger than $\mathbb{Q}$ but smaller than $\mathbb{R}$. This is a famous problem known as *the continuum hypothesis* and it is outside of the scope of this class! However, it is easy to construct an infinite set that is strictly larger that $\mathbb{R}$. Let A be the power set of $\mathbb{R}$, that is, A is the set of all subsets of R. The fact that A and $\mathbb{R}$ have different cardinal numbers can be shown by the same diagonal process argument that we used in part c) of exercise 1.1.

Exercise 2. a) $\mathcal{F} = \{\emptyset, \{2\}, \{5\}, \{1,3\}, \{2,5\}, \{4,6\}, \{1,2,3\}, \{1,3,5\}, \{2,4,6\}, \{4,5,6\},$ $\{1,2,3,5\}, \{1,3,4,6\}, \{2,4,5,6\}, \{1,2,3,4,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\}$ ($16 = 2^4$ elements)

b) atoms of $\mathcal{F}$: $\{1,3\}, \{2\}, \{4,6\}, \{5\}$. Notice that one also has $\mathcal{F} = \sigma(\{1,3\}, \{2\}, \{4,6\}, \{5\})$, as already mentioned in the problem set.

c) Nearly by definition, $\sigma(X_1, X_2) = \sigma(\{1,2,3\}, \{1,3,5\}) = \mathcal{F}$. Besides, the random variable Y satisfies: $Y(1) = Y(3) = 2$, $Y(2) = Y(5) = 1$ and $Y(4) = Y(6) = 0$. We deduce from there that the atoms of $\sigma(Y)$ are $\{1,3\}$, $\{2,5\}$ and $\{4,6\}$, and therefore that Y contains less information than X_1, X_2, i.e., that $\sigma(Y) \subset \sigma(X_1, X_2)$ and $\sigma(Y) \neq \sigma(X_1, X_2)$.

Exercise 3. Here is a systematic but not necessarily optimal procedure, described in words.

Consider first the list of subsets $\mathcal{L} = \{A_1, \ldots, A_m, A_1^c, \ldots, A_m^c\}$. From there, generate the list $\mathcal{L}' = \{B_1, \ldots, B_p\}$ made of all possible intersections of elements of $\mathcal{L}$ (which are *subsets* of Ω). Of course, this new list is not necessarily made of atoms of $\mathcal{F}$ only. We need to browse the collection and at each item, call it G, we discard it if it is empty or if there exists another element F in the collection such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$. The remaining elements are the atoms of $\mathcal{F}$.

Exercise 4. a) The atoms of $\mathcal{F}$ are the singletons $\{x\}$, with $x \in [0,1]$.

b) The answer is no. One can check indeed that the σ-field generated by the sets $\{x\}$, $x \in [0,1]$ is the list of all countable subsets of $[0,1]$, as well as all the complements of countable subsets of $[0,1]$, which is of course not equal to the list of all Borel subsets of $[0,1]$. In particular, the open intervals are not in the list.

c) $\sigma(\{x\}, x \in [0,1])$ comprises all countable unions of singletons in $[0,1]$, as well as all the complements of these sets. One can check that indeed, such a collection of sets is a σ-field, which is moreover *much* smaller than $\mathcal{B}([0,1])$.

Exercise 5. a) Consider e.g. X_1 taking values in $\{0,1\}$ and X_2 taking values in $\{0,2\}$. Then it is possible to deduce the values of both X_1 and X_2 from the sole value of Y, so $\sigma(Y) = \sigma(X_1, X_2)$ (as an exercise, write this down formally).

b) Consider e.g. X_1 taking values in $\{3,5\}$ and X_2 taking values in $\{7,9\}$. When $Y(\omega) = 12$, it is impossible to tell whether $X_1(\omega) = 3, X_2(\omega) = 9$ or $X_1(\omega) = 5, X_2(\omega) = 7$. The random variable Y carries then less information than the two random variables X_1, X_2 together (again, as an exercise, write this down formally).

c) The answer is no, i.e., $\sigma(Y) \neq \sigma(X_1, X_2)$, as when $Y(\omega) = a + b$, we will not be able to tell whether $\omega = \omega_1$ or $\omega = \omega_2$.

Exercise 6. a) Use $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint, as well as $\Omega = A \cup A^c$ and $\mathbb{P}(\Omega) = 1$.

b) Use $A \cup B = A \cup (B \backslash (A \cap B))$ where A and $B \backslash (A \cap B)$ are disjoint, as well as a).

c) Use $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$, where $B_n = A_n \backslash (A_1 \cup \ldots \cup A_{n-1})$; the B_n are disjoint, so by axiom (ii) and a),

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

d) $\mathbb{P}(\cup_{n\geq 1} A_n) = \mathbb{P}(\cup_{n\geq 1}(A_n \cap A_{n-1}^c)) \overset{(*)}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n \cap A_{n-1}^c) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{i-1}^c)$ $\overset{(**)}{=} \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n}(A_i \cap A_{i-1}^c)) = \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$, where $(*)$, $(**)$ follow from the fact that the sets $A_n \cap A_{n-1}^c$ are disjoint.

e) Using parts a) and d): $\mathbb{P}(\cap_{n\geq 1} A_n) = 1 - \mathbb{P}((\cap_{n\geq 1} A_n)^c) = 1 - \mathbb{P}(\cup_{n\geq 1} A_n^c) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^c) = \lim_{n\to\infty} \mathbb{P}(A_n)$.