Homework 1

Review. Recall that if there exists a 1-1 mapping of A *onto* B, we say that A and B have the same *cardinal number*, or, that A and B are equivalent, and we write $A \sim B$.
Definition. For any positive integer n, let $J_n = \{1, 2, \ldots, n\}$ be the set of first n positive integers and J be the set of all positive integers. For any A we say

(a) A is *finite* if $A \sim J_n$ for some n or $A = \emptyset$.

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is either finite or countable.

Exercise 1 Let $\mathbb{Q}$ denote rational numbers and $\mathbb{R}$ denote real numbers.

a) Show that $\mathbb{Q}$ is countable.

b) Show that every infinite subset of a countable set A is countable.

Remark: Roughly speaking, countable sets represent the "smallest" infinity.

c) Let the set A be all sequences whose elements are the digits 0 and 1. Show that A is not countable. Conclude that $\mathbb{R}$ is not countable.

d) Are irrational numbers, e.g. $\mathbb{R} \setminus \mathbb{Q}$, countable? Why or why not?

e) Construct a set that is infinite, but does not have the same cardinal number as $\mathbb{Q}$ or $\mathbb{R}$.

Exercise 2. Let $\Omega = \{1, \ldots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 3, 5\}\}$.

a) Describe $\mathcal{F} = \sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$.

Hint: For a finite set Ω, the number of elements of a σ-field on Ω is always a power of 2.

b) Give the list of non-empty elements G of $\mathcal{F}$ such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$

These elements are called the *atoms* of the σ-field $\mathcal{F}$ (cf. course). Equivalently, an event $G \in \mathcal{F}$ is *not* an atom if there exists $F \in \mathcal{F}$ such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$.

The atoms of a $\mathcal{F}$ form a *partition* of the set Ω and they also generate the σ-field $\mathcal{F}$ in this case. (note also that if m is the number of atoms of $\mathcal{F}$, then the number of elements of $\mathcal{F}$ equals 2^m)

c) Let $X_1(\omega) = 1_{\{1,2,3\}}(\omega)$, $X_2 = 1_{\{1,3,5\}}(\omega)$ and $Y(\omega) = X_1(\omega) + X_2(\omega)$. Does it hold that $\sigma(Y) = \sigma(X_1, X_2)$?

Exercise 3. Let $\Omega = \{1, \ldots, n\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of subsets of Ω. Describe a systematic method to find the list of atoms of the σ-field $\sigma(\mathcal{A})$.

Exercise 4. Let now $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}([0, 1])$ be the Borel σ-field on $[0, 1]$.

a) What are the atoms of $\mathcal{F}$?

b) Is it true in this case that the σ-field $\mathcal{F}$ is generated by its atoms?

c) Describe the σ-field $\sigma(\{x\}, x \in [0, 1])$.

Exercise 5*.

Let Ω be an arbitrary set and $\mathcal{F}$ be a $\sigma-$field on Ω. In this problem we will show that if $\mathcal{F}$ is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that $\mathcal{F}$ is countable.

a) For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}:\, \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?

b) Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω. Argue that $\mathcal{C}$ partitions Ω and that it is at most finite, or countable.

c) Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ-field generated by $\mathcal{C}$ is exactly $\mathcal{F}$.

d) Conclude from parts (a) - (c) that there is a contradiction and it is not possible for $\mathcal{F}$ to be countable.

Exercise 6. Let $\mathcal{F}$ be a σ-field on a set Ω and X_1, X_2 be two $\mathcal{F}$-measurable random variables taking a finite number of values in $\mathbb{R}$. Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subset \sigma(X_1, X_2)$, i.e., that X_1, X_2 carry together at least as much information as Y, but that the reciprocal statement is not necessarily true.

a) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) = \sigma(X_1, X_2)$.

b) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) \neq \sigma(X_1, X_2)$.

c) Assume that there exists $\omega_1 \neq \omega_2$ and $a \neq b$ such that $X_1(\omega_1) = X_2(\omega_2) = a$ and $X_1(\omega_2) = X_2(\omega_1) = b$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?