Homework 1

Review. Recall that if there exists a 1-1 mapping of A *onto* B, we say that A and B have the same *cardinal number*, or, that A and B are equivalent, and we write $A \sim B$.
Definition. For any positive integer n, let $J_n = \{1, 2, \ldots, n\}$ be the set of first n positive integers and J be the set of all positive integers. For any A we say

(a) A is *finite* if $A \sim J_n$ for some n or $A = \emptyset$.

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is either finite or countable.

Exercise 1.1.* Let $\mathbb{Q}$ denote rational numbers and $\mathbb{R}$ denote real numbers.

a) Show that $\mathbb{Q}$ is countable.

b) Show that every infinite subset of a countable set A is countable.

Remark: Roughly speaking, countable sets represent the "smallest" infinity.

c) Let the set A be all sequences whose elements are the digits 0 and 1. Show that A is not countable. Conclude that $\mathbb{R}$ is not countable.

d) Are irrational numbers, e.g. $\mathbb{R} \setminus \mathbb{Q}$, countable? Why or why not?

Exercise 1.2. Construct a set that is infinite, but does not have the same cardinal number as $\mathbb{Q}$ or $\mathbb{R}$.

Exercise 2. Let $\Omega = \{1, \ldots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 3, 5\}\}$.

a) Describe $\mathcal{F} = \sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$.

Hint: For a finite set Ω, the number of elements of a σ-field on Ω is always a power of 2.

b) Give the list of non-empty elements G of $\mathcal{F}$ such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$

These elements are called the *atoms* of the σ-field $\mathcal{F}$ (cf. course). Equivalently, an event $G \in \mathcal{F}$ is *not* an atom if there exists $F \in \mathcal{F}$ such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$.

The atoms of a $\mathcal{F}$ form a *partition* of the set Ω and they also generate the σ-field $\mathcal{F}$ in this case. (note also that if m is the number of atoms of $\mathcal{F}$, then the number of elements of $\mathcal{F}$ equals 2^m)

c) Let $X_1(\omega) = 1_{\{1,2,3\}}(\omega)$, $X_2 = 1_{\{1,3,5\}}(\omega)$ and $Y(\omega) = X_1(\omega) + X_2(\omega)$. Does it hold that $\sigma(Y) = \sigma(X_1, X_2)$?

Exercise 3. Let $\Omega = \{1, \ldots, n\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of subsets of Ω. Describe a systematic method to find the list of atoms of the σ-field $\sigma(\mathcal{A})$.

Exercise 4. Let now $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$ be the Borel σ-field on $[0,1]$.

a) What are the atoms of $\mathcal{F}$?

b) Is it true in this case that the σ-field $\mathcal{F}$ is generated by its atoms?

c) Describe the σ-field $\sigma(\{x\}, x \in [0,1])$.

Exercise 5. Let $\mathcal{F}$ be a σ-field on a set Ω and X_1, X_2 be two $\mathcal{F}$-measurable random variables taking a finite number of values in $\mathbb{R}$. Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subset \sigma(X_1, X_2)$, i.e., that X_1, X_2 carry together at least as much information as Y, but that the reciprocal statement is not necessarily true.

a) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) = \sigma(X_1, X_2)$.

b) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) \neq \sigma(X_1, X_2)$.

c) Assume that there exists $\omega_1 \neq \omega_2$ and $a \neq b$ such that $X_1(\omega_1) = X_2(\omega_2) = a$ and $X_1(\omega_2) = X_2(\omega_1) = b$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?

Exercise 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using only the axioms given in the definition of a probability measure, namely:

(i) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$;

(ii) If $(A_n,\, n \geq 1)$ is a sequence of *disjoint* events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$;

show the properties below.

Important note: For parts c), d) and e), induction does *not* work! You need to show that each property holds for an infinite number of events at once, using the above axiom (ii).

a) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$. Also, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

b) If $A, B \in \mathcal{F}$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

c) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

d) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \subset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

e) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.