Solutions 9

1. Because of the assumptions made, \( a_{ij} > 0 \) if \( \psi_{ij} > 0 \), so the chain with transition probabilities \( p_{ij} \) is also irreducible and aperiodic, therefore ergodic, as the state space \( S \) is finite. Let us check the detailed balance equation:

\[
\pi_i p_{ij} = \pi_i \psi_{ij} a_{ij} = \frac{\pi_i \psi_{ij} \pi_j \psi_{ji}}{\pi_j \psi_{ji} + \pi_i \psi_{ij}}
\]

which is clearly symmetric in \( i \) and \( j \), and therefore equal to \( \pi_j p_{ji} \).

2.a) The base chain must be irreducible and aperiodic, and such that \( \psi_{ij} > 0 \) if and only if \( \psi_{ji} > 0 \). There are of course many possible choices for \( \psi \). The symmetric random walk on \( \{0, \ldots, n\} \):

\[
\psi_{ij} = \begin{cases} 
1/2, & \text{if } |i - j| = 1 \\
1/2, & \text{if } i = j = 0 \text{ or } i = j = n \\
0, & \text{otherwise},
\end{cases}
\]

is a simple choice satisfying the above assumptions (since also \( \psi_{ij} = \psi_{ji} \)).

b) The acceptance probabilities are given by the formula \( a_{ij} = \min \left( 1, \frac{\pi_j}{\pi_i} \psi_{ji} \right) \), where \( \pi \) is the probability mass function of Bin\((n, p)\). Note that with our choice of base chain, for \( i, j \in \{0, \ldots, n\} \) such that \( i \neq j \), we have \( \psi_{ij} = \psi_{ji} \), so the acceptance probabilities become

\[
a_{ij} = \begin{cases} 
\min \left( 1, \frac{\pi_j}{\pi_i} \right), & \text{if } |i - j| = 1 \\
0, & \text{otherwise},
\end{cases}
\]

Computing these exactly using the probability mass function of the binomial distribution, we get

\[
a_{ij} = \begin{cases} 
\min \left( 1, \frac{\binom{n}{i+1}(1-p)^{n-i-1}}{\binom{i}{1}p(1-p)^{n-i}} \right) = \min \left( 1, \frac{(n-i)p}{(i+1)(1-p)} \right), & \text{if } j = i + 1 \\
\min \left( 1, \frac{\binom{n}{i-1}(1-p)^{n-i+1}}{\binom{i}{1}p(1-p)^{n-i}} \right) = \min \left( 1, \frac{i(1-p)}{(n-i+1)p} \right), & \text{if } j = i - 1 \\
0, & \text{otherwise},
\end{cases}
\]

Note that because of the presence of the ratio \( \pi_j/\pi_i \), we observe here a simplification in the expression for \( a_{ij} \), where the factorials have disappeared. This is similar to what happens with the normalization constant \( Z \) in other situations where the Metropolis algorithm is applied.
c) The matrix $P$ is given by

$$p_{ij} = \begin{cases} \psi_{ij} a_{ij}, & \text{if } |i-j| = 1 \\ 1 - \frac{1}{2} a_{ij}, & \text{if } |i-j| = 1 \\ 1 - \frac{1}{2}(a_{i,i+1} + a_{i,i-1}), & \text{if } i = j \end{cases}$$

Since we used the Metropolis-Hastings procedure, it is guaranteed that the stationary distribution $\pi$ of the chain $P$ is the Bin$(n,p)$ distribution.

d) In the following, we denote the Markov process induced by chain $P$ by $X_i$ at time step $i$.

In order to get the samples, we would start the chain $P$ in some state, say $X_0 = 0$. We would then perform a random walk by applying powers $P^n$ large $n$, and return as sample $X_n$.

However, to have faithful samples, we need that the distribution of $X_n$ converges to the stationary distribution $\pi$, which is true only if $\pi$ is also the limiting distribution. This happens if $P$ is ergodic.

We check indeed that $P$ is ergodic (but note that we know this by the theorem seen in class). Since the number of states is finite, we only need to verify that $P$ is irreducible and aperiodic.

1. **Irreducibility**: Looking at the acceptance probabilities from part b), we see that whenever $|i-j| = 1$, we have that $a_{ij} > 0$. Let us look at the case $j = i+1$ (the case $j = i - 1$ is similar). We note that the fraction $\frac{p}{1-p}$ is always in the interval $(0, +\infty)$ since $0 < p < 1$, and the fraction $\frac{n-i}{i+1}$ is also a positive number. Thus $a_{i,i+1} = \min\left(1, \frac{(n-i)p}{(i+1)(1-p)}\right) > 0$. In the same manner, we get $a_{i,i-1} > 0$.

Now using part c), this means in turn that $P_{ij} = \frac{1}{2} a_{ij} > 0$ when $|i-j| = 1$. Thus, all states communicate and $P$ is irreducible.

2. **Aperiodicity**: If there are self-loops, we are done. We show that it is the case for state 0, indeed we have $P_{00} = 1 - \frac{1}{2}a_{01} = 1 - \frac{1}{2} \min(1, \frac{np}{1-p}) \geq \frac{1}{2} > 0$. The chain $P$ is thus aperiodic.

We conclude that $P$ is indeed an ergodic chain, and thus as $n$ gets large, the distribution of $X_n$ will converge to the desired distribution $\pi$.

3.a) There is no stationary distribution. By computing the stationary distribution, one would find an infinite normalization constant. The walk is irreducible but since there is no stationary distribution it cannot be positive-recurrent.

b) The acceptance probability for the move $i \to i + 1$ is

$$a_{i,i+1} = \min(1, e^{-a((i+1)^2-i^2)}) = \begin{cases} 1, & i \leq -1 \\ e^{-a(2i+1)}, & i \geq 0. \end{cases}$$

and for the move $i \to i - 1$ is

$$a_{i,i-1} = \min(1, e^{-a((i-1)^2-i^2)}) = \begin{cases} 1, & i \geq 1 \\ e^{a(2i-1)}, & i \leq 0. \end{cases}$$
In words: a move towards the origin \( i = 0 \) is always accepted and a move away from the origin is accepted with probability \( e^{-a(1+2i)} \).

The transition probabilities are

\[
p_{i,i+1} = \frac{1}{2} \min(1, e^{-a((i+1)^2-i^2)}) = \begin{cases} 
\frac{1}{2}, & i \leq -1, \\
\frac{1}{2} e^{-a(2i+1)}, & i \geq 0.
\end{cases}
\]

and

\[
p_{i,i-1} = \frac{1}{2} \min(1, e^{-a((i-1)^2-i^2)}) = \begin{cases} 
\frac{1}{2}, & i \geq 1, \\
\frac{1}{2} e^{a(i-1)}, & i \leq 0.
\end{cases}
\]

and for the self-loops

\[
p_{ii} = 1 - p_{i,i-1} - p_{i,i+1} = \begin{cases} 
1 - \frac{1}{2} e^{a(2i-1)} - \frac{1}{2} = \frac{1}{2} (1 - e^{a(2i-1)}), & i \leq -1, \\
1 - \frac{1}{2} e^{-a} - \frac{1}{2} e^{-a} = 1 - e^{-a}, & i = 0, \\
1 - \frac{1}{2} - \frac{1}{2} e^{-a(2i+1)} = \frac{1}{2} (1 - e^{-a(2i+1)}), & i \geq 1.
\end{cases}
\]

c) The walk is obviously irreducible. A stationary distribution \( \pi^* \) exists by construction. Thus by the first fundamental theorem in class the walk is positive recurrent. Moreover because of the self loops it is aperiodic. An irreducible, positive recurrent, aperiodic walk satisfies the ergodic theorem and therefore

\[
\lim_{n \to +\infty} (P^n)_{ij} = \pi^*_j
\]

where \( P \) is the transition matrix elements \((P^n)_{ij} = \mathbb{P}(X_n = j \mid X_0 = i)\).

d) We start on the right of the origin because \( z > 0 \). The initial distance between the two walkers is \( d \). When we propose a move towards the left \( \xi_1 = -1 \) both walkers accept the move with probability 1 so the distance stays equal to \( d \). When we propose a move towards the right \( \xi_1 = +1 \) the walkers may accept-accept the move, accept-reject the move, reject-accept the move, or reject-reject the move. In the first case and in the last case the distance stays equal to \( d \), in the second case it decreases to \( d - 1 \) and the third case it increases to \( d + 1 \). Reasoning like this, we see that the minimum coalescence time corresponds to the events where we always propose a right move and the walkers accept-reject.

This minimum coalescence time is thus equal to \( d \). We have

\[
\mathbb{P}(T = d) = \frac{1}{2^d} \left( \prod_{i=z}^{z+d-1} e^{-a(i+1)} \right) (1 - e^{-a(2(z+d)+1)})^d
\]