Solutions 13

1. a) The transition probabilities are given by the following formula:

$$p(x,y) = \begin{cases} \dfrac{\pi(x'_u \mid x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d}, & \text{if } y = (x_1, \cdots, x_{u-1}, x'_u, x_{u+1}, \cdots, x_d) \\ & \text{for } u \in \{1, \cdots, d\} \text{ and } x_u \neq x'_u, \\ \dfrac{\sum_{u=1}^{d} \pi(x_u | x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d}, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

b) Let us now show that the detailed balance equation is satisfied.

First note from (1) that $p(x,y) = 0$ if and only if x and y differ in more than a single component. Hence, $p(x,y) = 0$ if and only if $p(y,x) = 0$ and the detailed balance equation is satisfied since both sides are zero.

Then, suppose that x and y differ in component u, i.e., $x = (x_1, \cdots, x_{u-1}, x_u, x_{u+1}, \cdots, x_d)$ and $y = (x_1, \cdots, x_{u-1}, x'_u, x_{u+1}, \cdots, x_d)$. Therefore, from (1) we have

$$\begin{aligned} \pi(x)p(x,y) &= \frac{\pi(x)\pi(x'_u \mid x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d} = \frac{g(x)g(y)}{Z \cdot \sum_{y_u \in S} g(x_1, \ldots, x_{u-1}, y_u, x_{u+1}, \ldots, x_d)} \\ &= \pi(y)p(y,x). \end{aligned}$$

As a result, this algorithm can be viewed as a Metropolis-Hastings algorithm where the base chain has the transition probabilities $p(x,y)$ defined in (1) and the acceptance probability $a(x,y) = 1$ for any $x, y \in S^d$. In words, every move is always accepted. Note that every base chain which satisfies the detailed balance equation induces a Metropolis Hastings algorithm in which every move is always accepted. To see this, consider the base chain ψ_{ij} s.t. $\pi_i \psi_{ij} = \pi_j \psi_{ji}$ for any i, j in some state space $\mathcal{S}$. Then, as the base chain is not necessarily symmetric, the acceptance probability is given by

$$a_{ij} = \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right) = 1.$$