Markov Chains and Algorithmic Applications - IC - EPFL

Solutions 1

1. Using Stirling’s approximation for \( \binom{2n}{n} \), we obtain
   \[
   \binom{2n}{n} p^n q^n \sim \frac{2\pi(2n)^{2n}}{2\pi n \left(\frac{n}{e}\right)^{2n}} (pq)^n = \frac{(4pq)^n}{\sqrt{\pi n}}
   \]

2. a) Both \( X \) and \( Y \) are random walks with probability 1/4 to go in either direction, and probability 1/2 to stay in place.
   
b) No, they are not independent: when \( X \) makes a move, \( Y \) does not, and vice-versa.
   
c) Both \( U \) and \( V \) are simple symmetric random walks with probability 1/2 to go in either direction.
   
d) Yes, they are independent. Denote \( U_n = \eta_1 + \ldots + \eta_n, V_n = \chi_1 + \ldots + \chi_n \). Then one can check e.g. that (and similarly for all \( \pm 1 \) combinations)
   \[
   \mathbb{P}(\eta_n = +1, X_n = +1) = \mathbb{P}(\eta_n = +1) = \frac{1}{4} = \mathbb{P}(\eta_n = +1) \cdot \mathbb{P}(X_n = +1)
   \]
   
e) Note that \( \overrightarrow{S_{2n}} = (0, 0) \) if and only if \( U_{2n} = V_{2n} = 0 \), so by the independence shown above, we obtain
   \[
   \mathbb{P}\left(\overrightarrow{S_{2n}} = (0, 0) \mid \overrightarrow{S_0} = (0, 0)\right) = \mathbb{P}(U_{2n} = 0, V_{2n} = 0 \mid U_0 = 0, V_0 = 0)
   \]
   \[
   = \mathbb{P}(U_{2n} = 0 \mid U_0 = 0) \cdot \mathbb{P}(V_{2n} = 0 \mid V_0 = 0) = \left(\binom{2n}{n} 2^{-2n}\right)^2 \sim \frac{1}{\pi n}
   \]
   by Exercise 1.

3. Consider \( i \) and \( j \) are two intercommunicating states. For arbitrary \( m, n, \) and \( r \in \mathbb{N} \), we have
   \[
   p_{ii}^{(m+n+r)} = \mathbb{P}(X_{m+n+r} = i \mid X_0 = i) = \sum_{k_1, k_2} \mathbb{P}(X_{m+n+r} = i, X_{m+r} = k_2, X_m = k_1 \mid X_0 = i)
   \]
   \[
   = \sum_{k_1, k_2} \mathbb{P}(X_{m+n+r} = i \mid X_{m+r} = k_2) \mathbb{P}(X_{m+r} = k_2 \mid X_m = k_1) \mathbb{P}(X_m = k_1 \mid X_0 = i)
   \]
   which can be rewritten as
   \[
   p_{ii}^{(m+n+r)} = \sum_{k_1, k_2} p_{k_2}^{(r)} p_{k_1}^{(m)} p_{ij}^{(n)} \geq p_{ji}^{(n)} p_{ij}^{(r)} p_{ij}^{(m)}
   \]
   Since \( i \) and \( j \) are intercommunicating states, there always exist \( m \) and \( n \in \mathbb{N} \) such that \( p_{ij}^{(m)} > 0 \) and \( p_{ji}^{(n)} > 0 \). So, let us consider \( n \) and \( m \) fixed, and define \( \alpha = p_{ji}^{(n)} p_{ij}^{(m)} > 0 \). The inequality then can be rewritten as a function of \( \alpha \):
   \[
   p_{ii}^{(m+n+r)} \geq \alpha p_{jj}^{(r)}
   \]
   Therefore, \( p_{jj}^{(r)} \) can be non-zero only if \( p_{ii}^{(m+n+r)} \) is non-zero. \( p_{ii}^{(m+n+r)} \) is non-zero only if \( d(i) | m + n + r \). At the same time, for the case \( r = 0 \), we have \( p_{ii}^{(m+n)} \geq \alpha > 0 \), which means that \( d(i) | m + n \). Therefore, \( p_{jj}^{(r)} \) can be non-zero only if \( d(i) | r \), which means that \( d(i) | d(j) \). With the same argument, we have \( d(j) | d(i) \), and as a conclusion we have \( d(j) = d(i) \).