Exercise 1. Let \((S_n, n \in \mathbb{N})\) be the simple asymmetric random walk on \(\mathbb{Z}\), defined as
\[
S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1,
\]
where the random variables \((\xi_n, n \geq 1)\) are i.i.d. with \(P(\xi_n = +1) = p \in ]0, 1[\) and \(P(\xi_n = -1) = q = 1 - p\). Using Stirling’s formula (valid for large values of \(n\)):
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,
\]
show that
\[
P_{0,0}^{(2n)} = P(S_{2n} = 0 | S_0 = 0) \sim \frac{(4pq)^n}{\sqrt{\pi n}}.
\]

NB: The notation \(a_n \sim b_n\) means precisely
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

Exercise 2. Let \(\left(\vec{S}_n, n \in \mathbb{N}\right)\) be the simple symmetric random walk in two dimensions, that is,
\[
\vec{S}_0 = (0, 0), \quad \vec{S}_n = \vec{\xi}_1 + \ldots + \vec{\xi}_n, \quad n \geq 1,
\]
where \(\left(\vec{\xi}_n, n \geq 1\right)\) are i.i.d random variables such that
\[
P\left(\vec{\xi}_n = (1, 0)\right) = P\left(\vec{\xi}_n = (-1, 0)\right) = P\left(\vec{\xi}_n = (0, 1)\right) = P\left(\vec{\xi}_n = (0, -1)\right) = \frac{1}{4}.
\]
Let us write \(\vec{S}_n = (X_n, Y_n)\).

a) Compute the transition matrices of the random walks \((X_n, n \in \mathbb{N})\) and \((Y_n, n \in \mathbb{N})\).

b) Are these two random walks independent?

Define now \(U_n = X_n + Y_n\) and \(V_n = X_n - Y_n, n \in \mathbb{N}\). Again the same questions:

c) Again, compute the transition matrices of the random walks \((U_n, n \in \mathbb{N})\) and \((V_n, n \in \mathbb{N})\).

d) Are these two random walks independent?

e) Deduce from this the value of \(P\left(\vec{S}_{2n} = (0, 0) | \vec{S}_0 = (0, 0)\right)\). How does it behave for large \(n\)?
Exercise 3. Prove that the intercommunicating states of a Markov chain have the same period.

Hint 1: Consider two intercommunicating states, $i$ and $j$. Then, find a lower bound for $p^{(m+n+r)}_{ii}$ as a function of $p^{(m)}_{ij}$, $p^{(n)}_{ji}$, and $p^{(r)}_{jj}$.

Hint 2: Show that $p^{(r)}_{jj}$ can be non-zero only if $d(i)|r$. Then, find an argument to conclude that $d(i) = d(j)$.

Note: $d|r$ is the notation for “$d$ divides $r$.”