Problem 1: Review of Random Variables

Let $X$ and $Y$ be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x, y)$. Let $a, b \in \mathbb{R}$ be fixed.

(a) Prove that $E[aX + bY] = aE[X] + bE[Y]$. Do not assume independence.

(b) Prove that if $X$ and $Y$ are independent random variables, then $E[X \cdot Y] = E[X] \cdot E[Y]$.

(c) Assume that $X$ and $Y$ are not independent. Find an example where $E[X \cdot Y] \neq E[X] \cdot E[Y]$, and another example where $E[X \cdot Y] = E[X] \cdot E[Y]$.

(d) Prove that if $X$ and $Y$ are independent, then they are also uncorrelated, i.e.,

$$\text{Cov}(X,Y) := E[(X - E[X])(Y - E[Y])] = 0.$$  \hfill (1)

(e) Find an example where $X$ and $Y$ are uncorrelated but dependent.

(f) Assume that $X$ and $Y$ are uncorrelated and let $\sigma_X^2$ and $\sigma_Y^2$ be the variances of $X$ and $Y$, respectively. Find the variance of $aX + bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$.

Hint: First show that $\text{Cov}(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y]$.

Problem 2: Review of Gaussian Random Variables

A random variable $X$ with probability density function

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$ \hfill (2)

is called a Gaussian random variable.

(a) Explicitly calculate the mean $E[X]$, the second moment $E[X^2]$, and the variance $\text{Var}[X]$ of the random variable $X$.

(b) Let us now consider events of the following kind:

$$\text{Pr}(X < \alpha).$$ \hfill (3)

Unfortunately for Gaussian random variables this cannot be calculated in closed form. Instead, we will rewrite it in terms of the standard Q-function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du$$ \hfill (4)
Express \( \Pr(X < \alpha) \) in terms of the Q-function and the parameters \( m \) and \( \sigma^2 \) of the Gaussian pdf.

Like we said, the Q-function cannot be calculated in closed form. Therefore, it is important to have bounds on the Q-function. In the next 3 subproblems, you derive the most important of these bounds, learning some very general and powerful tools along the way:

(c) Derive the Markov inequality, which says that for any non-negative random variable \( X \) and positive \( a \), we have

\[
\Pr(X \geq a) \leq \frac{E[X]}{a} \tag{5}
\]

(d) Use the Markov inequality to derive the Chernoff bound: the probability that a real random variable \( Z \) exceeds \( b \) is given by

\[
\Pr(Z \geq b) \leq E[e^{s(Z-b)}], \quad s \geq 0. \tag{6}
\]

(e) Use the Chernoff bound to show that

\[
Q(x) \leq e^{-\frac{x^2}{2}} \quad \text{for } x \geq 0. \tag{7}
\]

**Problem 3: Moment Generating Function**

In the class we had considered the logarithmic moment generating function

\[
\phi(s) := \ln E[\exp(sX)] = \ln \sum_x p(x) \exp(sx)
\]

of a real-valued random variable \( X \) taking values on a finite set, and showed that \( \phi'(s) = E[X_s] \) where \( X_s \) is a random variable taking the same values as \( X \) but with probabilities \( p_s(x) := p(x) \exp(sx) \exp(-\phi(s)) \).

(a) Show that

\[
\phi''(s) = \text{Var}(X_s) := E[X_s^2] - E[X_s]^2
\]

and conclude that \( \phi''(s) \geq 0 \) and the inequality is strict except when \( X \) is deterministic.

(b) Let \( x_{\text{min}} := \min\{x : p(x) > 0\} \) and \( x_{\text{max}} := \max\{x : p(x) > 0\} \) be the smallest and largest values \( X \) takes. Show that

\[
\lim_{s \to -\infty} \phi'(s) = x_{\text{min}}, \quad \text{and} \quad \lim_{s \to \infty} \phi'(s) = x_{\text{max}}.
\]
Problem 4: Hoeffding’s Lemma

Prove Lemma 2.3 in the lecture notes. In other words, prove that if $X$ is a zero-mean random variable taking values in $[a, b]$ then

$$E[e^{\lambda X}] \leq e^{\frac{\lambda^2(a-b)^2}{4}}.$$  

Expressed differently, $X$ is $[(a-b)^2/4]$-subgaussian.

*Hint:* You can use the following steps to prove the lemma:

1. Let $\lambda > 0$. Let $X$ be a random variable such that $a \leq X \leq b$ and $E[X] = 0$. By considering the convex function $x \rightarrow e^{\lambda x}$, show that

$$E[e^{\lambda X}] \leq \frac{b}{b-a} e^{\lambda a} - \frac{a}{b-a} e^{\lambda b}. \tag{8}$$

2. Let $p = -a/(b-a)$ and $h = \lambda(b-a)$. Verify that the right-hand side of (8) equals $e^{L(h)}$ where

$$L(h) = -hp + \log(1 - p + pe^h).$$

3. By Taylor’s theorem, there exists $\xi \in (0, h)$ such that

$$L(h) = L(0) + hL'(0) + \frac{h^2}{2}L''(\xi).$$

Show that $L(h) \leq h^2/8$ and hence $E[e^{\lambda X}] \leq e^{\lambda^2(b-a)^2/8}$.

Problem 5: Expected Maximum of Subgaussians

Let $\{X_i\}_{i=1}^n$ be a collection of $n$ $\sigma^2$-subgaussian random variables, not necessarily independent of each other. Let $Y = \max_{i \in \{1, 2, \ldots, n\}} X_i$. Prove that $E[Y] \leq \sqrt{2\sigma^2 \log n}$. *Hint:* Recall that by Jensen, $e^{\lambda E[X]} \leq E[e^{\lambda X}]$.  

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