Homework 4

Exercise 1. a) Let μ and ν be two distributions on a state space S (i.e., μ_i, $\nu_i \geq 0$ for every $i \in S$ and $\sum_{i\in S} \mu_i = \sum_{i\in S} \nu_i = 1$). Show that the following three definitions of the total variation distance between μ and ν are equivalent:

1. $\|\mu - \nu\|_{\text{TV}} = \frac{1}{2} \sum_{i\in S} |\mu_i - \nu_i|$.
2. $\|\mu - \nu\|_{\text{TV}} = \sup_{A\subset S} |\mu(A) - \nu(A)|$, where $\mu(A) = \sum_{i\in A} \mu_i$ and $\nu(A) = \sum_{i\in A} \nu_i$.
3. $\|\mu - \nu\|_{\text{TV}} = \frac{1}{2} \sup_{\phi:S\to[-1,+1]} |\mu(\phi) - \nu(\phi)|$, where $\mu(\phi) = \sum_{i\in S} \mu_i\phi_i$ and $\nu(\phi) = \sum_{i\in S} \nu_i\phi_i$.

Hint: The easiest way is to show that $1 \leq 2 \leq 3 \leq 1$.

On $\mathcal{S} = \{0, 1\}$, let now μ, ν be the two distributions defined as $\mu = (3/4, 1/4)$ and $\nu = (1/4, 3/4)$.

b) Describe three different couplings (X, Y) of μ and ν such that: **b1)** X and Y are positively correlated, **b2)** X and Y are independent; **b3)** X and Y are negatively correlated.

c) Describe the coupling (X, Y) such that $\|\mu - \nu\|_{\text{TV}} = \mathbb{P}(X \neq Y)$.

NB: One can show that such a coupling always exists.

Exercise 2. Let $(X_n,\, n \geq 1)$ be a Markov chain with state space $\mathcal{S} = \{0, 1\}$, initial distribution $\pi^{(0)}$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad \text{where} \quad 0 < p, q < 1$$

Let $(Y_n, n \geq 1)$ be another Markov chain with same state space $\mathcal{S}$ and same transition matrix P, but whose initial distribution π is also the stationary distribution of the chain.

a) Compute π.

We consider now the coupled chain $Z = (X, Y)$ with state space $\mathcal{S} \times \mathcal{S}$ such that X, Y evolve independently according to P as long as $X_n \neq Y_n$, and then evolve together, still according to P, as soon as $X_n = Y_n$.

b) Write down the transition matrix P_Z of the chain Z.

c) Which states of Z are transient / recurrent?

d) Does the chain Z admit a unique limiting and stationary distribution π_Z? If yes, compute it; if no, explain why.

e) Express $\mathbb{P}(X_{n+1} \neq Y_{n+1})$ as a function of $\mathbb{P}(X_n \neq Y_n)$.

f) From e), deduce an upper bound on $\max_{i\in\mathcal{S}} \|P_i^n - \pi\|_{\text{TV}}$.

g) When $p = q$, what value of $0 < p < 1$ leads to the fastest convergence?

Exercise 3. Let $(X_n, n \geq 0)$ be a time-homogeneous Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$. Moreover suppose that the Markov chain evolves according to the transition graph depicted in figure 1, where from any state, we move to a neighbouring state uniformly at random.

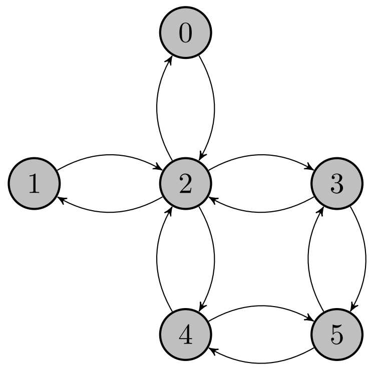

Figure 1: State diagram for the Markov chain $(X_n, n \geq 0)$

a) Find the transition matrix P associated to the chain.

b) Is the chain irreducible? Is it periodic or aperiodic? Is it ergodic?

c) Find its stationary distribution. Is it also a limiting distribution?

d) Starting from state 2, what is the expected number of steps before we come back to that state?

e) Compute the 2-step transition probabilities $p_{ij}^{(2)}$ for $i, j \in \mathcal{S}$.

f) Describe the equivalence classes of the chain with transition matrix $Q = P^2$.

g) Compute both $\lim_{n\to\infty} \mathbb{P}(X_n = 1 | X_0 = 1)$ and $\lim_{n\to\infty} \mathbb{P}(X_{2n} = 1 | X_0 = 1)$.