Exercise 1. a) Let $\mu$ and $\nu$ be two distributions on a state space $S$ (i.e., $\mu_i, \nu_i \geq 0$ for every $i \in S$ and $\sum_{i \in S} \mu_i = \sum_{i \in S} \nu_i = 1$). Show that the following three definitions of the total variation distance between $\mu$ and $\nu$ are equivalent:

1. $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i|.$

2. $\|\mu - \nu\|_{TV} = \sup_{A \subset S} |\mu(A) - \nu(A)|$, where $\mu(A) = \sum_{i \in A} \mu_i$ and $\nu(A) = \sum_{i \in A} \nu_i.$

3. $\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{\phi : S \rightarrow [-1, 1]} |\mu(\phi) - \nu(\phi)|$, where $\mu(\phi) = \sum_{i \in S} \mu_i \phi_i$ and $\nu(\phi) = \sum_{i \in S} \nu_i \phi_i.$

Hint: The easiest way is to show that $1 \leq 2 \leq 3 \leq 1.$

On $S = \{0, 1\}$, let now $\mu, \nu$ be the two distributions defined as $\mu = (3/4, 1/4)$ and $\nu = (1/4, 3/4)$.

b) Describe three different couplings $(X, Y)$ of $\mu$ and $\nu$ such that: \textbf{b1)} $X$ and $Y$ are positively correlated, \textbf{b2)} $X$ and $Y$ are independent; \textbf{b3)} $X$ and $Y$ are negatively correlated.

c) Describe the coupling $(X, Y)$ such that $\|\mu - \nu\|_{TV} = \mathbb{P}(X \neq Y)$.

NB: One can show that such a coupling always exists.

Exercise 2. Let $(X_n, n \geq 1)$ be a Markov chain with state space $S = \{0, 1\}$, initial distribution $\pi^{(0)}$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad \text{where} \quad 0 < p, q < 1$$

Let $(Y_n, n \geq 1)$ be another Markov chain with same state space $S$ and same transition matrix $P$, but whose initial distribution $\pi$ is also the stationary distribution of the chain.

a) Compute $\pi$.

We consider now the coupled chain $Z = (X, Y)$ with state space $S \times S$ such that $X, Y$ evolve independently according to $P$ as long as $X_n \neq Y_n$, and then evolve together, still according to $P$, as soon as $X_n = Y_n$.

b) Write down the transition matrix $P_Z$ of the chain $Z$.

c) Which states of $Z$ are transient / recurrent?

d) Does the chain $Z$ admit a unique limiting and stationary distribution $\pi_Z$? If yes, compute it; if no, explain why.

e) Express $\mathbb{P}(X_{n+1} \neq Y_{n+1})$ as a function of $\mathbb{P}(X_n \neq Y_n)$.

f) From e), deduce an upper bound on $\max_{i \in S} \|P^n_i - \pi\|_{TV}$.

g) When $p = q$, what value of $0 < p < 1$ leads to the fastest convergence?
Exercise 3. Let \((X_n, n \geq 0)\) be a time-homogeneous Markov chain with state space \(\mathcal{S} = \{0, 1, 2, 3, 4, 5\}\). Moreover suppose that the Markov chain evolves according to the transition graph depicted in figure 1, where from any state, we move to a neighbouring state uniformly at random.

![State diagram for the Markov chain \((X_n, n \geq 0)\)](image)

Figure 1: State diagram for the Markov chain \((X_n, n \geq 0)\)

a) Find the transition matrix \(P\) associated to the chain.

b) Is the chain irreducible? Is it periodic or aperiodic? Is it ergodic?

c) Find its stationary distribution. Is it also a limiting distribution?

d) Starting from state 2, what is the expected number of steps before we come back to that state?

e) Compute the 2-step transition probabilities \(p_{ij}^{(2)}\) for \(i, j \in \mathcal{S}\).

f) Describe the equivalence classes of the chain with transition matrix \(Q = P^2\).

g) Compute both \(\lim_{n \to \infty} \mathbb{P}(X_n = 1|X_0 = 1)\) and \(\lim_{n \to \infty} \mathbb{P}(X_{2n} = 1|X_0 = 1)\).