Exercise 1. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$ for every $n \geq 1$. Let also $(S_n, n \geq 0)$ the simple symmetric random walk defined as $S_0 = 0$, $S_{n+1} = S_n + X_{n+1}$ for $n \geq 0$.

Among the following discrete-time stochastic processes, which are (time-homogeneous) Markov chains, which are not? If a process is a Markov chain, prove it and compute its transition matrix $P_1$. If a process is not, find a counter-example showing that the Markov property is not satisfied (Hint: Consider small values of $n$!)

a) $Y_n = S_{2n}$ for $n \geq 0$.

b) $Z_n = (-1)^{S_n}$ for $n \geq 0$.

c) $T_n = \max\{S_0, S_1, \ldots, S_n\}$ for $n \geq 0$.

d) $W_0 = 0$ and $W_{n+1} = W_n + X_{2n+1} + X_{2n+2}$ for $n \geq 0$.

Exercise 2. Let $(X_n, n \geq 0)$ be a time-homogeneous Markov chain with state space $\mathcal{S} = \{0, 1\}$ and transition probabilities

$$
\mathbb{P}(X_{n+1} = 1 \mid X_n = 0) = p \quad \text{and} \quad \mathbb{P}(X_{n+1} = 0 \mid X_n = 1) = q
$$

where $0 < p, q < 1$.

a) Write down the transition matrix $P$ of the chain and draw its transition graph.

b) For which values of $p, q$ does it hold that $(X_n, n \geq 0)$ is a sequence of independent random variables?

c) Compute the $n$-step transition probabilities $p^{(n)}_{00} = \mathbb{P}(X_n = 0 \mid X_0 = 0)$.

Hint: Use the eigenvalue-eigenvector decomposition of the matrix $P$.

d) Compute $\sum_{n \geq 1} p^{(n)}_{00}$. Is it finite or not? What does that imply?

e) Let now $T_0 = \inf\{n \geq 1 : X_n = 0\}$. Compute $f^{(n)}_{00} = \mathbb{P}(T_0 = n \mid X_0 = 0)$ and $f_{00} = \mathbb{P}(T_0 < +\infty \mid X_0 = 0)$. Is your result coherent with what you have obtained in question d)?

f) Compute $\mathbb{E}(T_0 \mid X_0 = 0)$. Is it finite or not? What does that imply?

g) For questions e) and f), consider the following special cases: 1) $p + q = 1$ and 2) $p = q$. Interpret your results.
Exercise 3. Let \((X_n, n \in \mathbb{N})\) be a time-homogeneous Markov chain with \(n\)-step transition probabilities

\[
p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i)
\]

a) Using the criterion for recurrence seen in the lectures, show that in a given equivalence class, either all states are recurrent, or all states are transient.

We define now the probability of first passage as the probability that the chain passes from \(i\) to \(j\) in \(n\) steps without passing by \(j\) before the \(n^{th}\) step:

\[
f_{ij}^{(n)} = \mathbb{P}(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)
\]

Note: When \(i = j\), this matches the definition of \(f_{ii}^{(n)}\) seen in class. Let also

\[
P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n \quad p_{ij}(0) = \delta_{ij}
\]

\[
F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n \quad f_{ij}(0) = 0
\]

be the associated generating functions. Recall that we proved in class that \(P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s)\).

b) Prove that for \(i \neq j\):

\[
P_{ij}(s) = F_{ij}(s) P_{jj}(s)
\]

c) Deduce the following statements:

1. If \(j\) is recurrent, then \(\sum_{n \geq 0} p_{ij}^{(n)} = +\infty\) for all \(i\) such that \(f_{ij} > 0\), where \(f_{ij} = \sum_{n \geq 0} f_{ij}^{(n)}\).

2. If \(j\) is transient, then \(\sum_{n \geq 0} p_{ij}^{(n)} < +\infty\) for all \(i\).