Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ be an square-integrable random variable defined on this space and let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If $X$ is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If $X$ is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If $Y$ is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})Y$ a.s.

e) If $\mathcal{H}$ is a sub-$\sigma$-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Hint for parts b) to e): According to the course definition, in order to check that some candidate random variable $Z$ is the conditional expectation of $X$ given $\mathcal{G}$, you should check the following two conditions:

(i) $Z \in \mathcal{G}$, i.e., $Z$ is $\mathcal{G}$-measurable and square-integrable;

(ii) $Z$ satisfies $\mathbb{E}((Z - X)U) = 0$ for every $U \in \mathcal{G}$.

Exercise 2. Let $X$, $Y$ be two discrete random variables (with values in a countable set $C$). Let us moreover assume that $X$ is square-integrable.

a) Show that the random variable $\psi(Y)$, where $\psi$ is defined as

$$\psi(y) = \sum_{x \in C} x \ \mathbb{P}(\{X = x\}|\{Y = y\})$$

matches the definition of conditional expectation $\mathbb{E}(X|Y)$ given in the lectures.

b) Application: One rolls two independent and balanced dice (say $Y$ and $Z$), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Exercise 3. Let $X$ be a random variable such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Z \sim \mathcal{N}(0,1)$ be independent of $X$. Let also $a > 0$ and $Y = aX + Z$. We propose below four possible estimators of the variable $X$ given the noisy observation $Y$:

$$\hat{X}_1 = \frac{Y}{a}, \quad \hat{X}_2 = \frac{aY}{a^2 + 1}, \quad \hat{X}_3 = \text{sign}(aY), \quad \hat{X}_4 = \tanh(aY)$$

a) Which estimator among these four minimizes the mean square error (MSE) $\mathbb{E}((\hat{X} - X)^2)$? In order to answer the question, draw on the same graph the four curves representing the MSE as a function of $a > 0$. For this, you may use either the exact mathematical expression of the MSE or the one obtained via Monte-Carlo simulations.

b) Provide a theoretical justification for your conclusion.

c) For which of the four estimators above does it hold that $\mathbb{E}((\hat{X} - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\hat{X}^2)$?