Exercise 1. (extended law of large numbers)

Let \((\mu_n, n \geq 1)\) be a sequence of real numbers such that

\[
\lim_{n \to \infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \mu \in \mathbb{R}
\]

Let \((X_n, n \geq 1)\) be a sequence of square-integrable random variables such that

\[
E(X_n) = \mu_n, \quad \forall n \geq 1 \quad \text{and} \quad \text{Cov}(X_n, X_m) \leq C_1 \exp(-C_2 |m - n|), \quad \forall m, n \geq 1.
\]

for some constants \(C_1, C_2 > 0\) (the random variables \(X_n\) are said to be weakly correlated). Let finally \(S_n = X_1 + \ldots + X_n\).

a) Show that

\[
\frac{S_n}{n} \xrightarrow{p} \frac{\mu}{n \to \infty}
\]

b) Is it also true that

\[
\frac{S_n}{n} \xrightarrow{a.s.} \mu \quad \text{almost surely?}
\]

In order to check this, you need to go through the proof of the strong law of large numbers made in class. Does that proof need the fact that the random variables \(X_n\) are independent?

c) Application: Let \((Z_n, n \geq 1)\) be a sequence of i.i.d. \(\sim \mathcal{N}(0, 1)\) random variables, \(x, a \in \mathbb{R}\) and \((X_n, n \geq 1)\) be the sequence of random variables defined recursively as

\[
X_1 = x, \quad X_{n+1} = aX_n + Z_{n+1}, \quad n \geq 1
\]

For what values of \(x, a \in \mathbb{R}\) does the sequence \((X_n, n \geq 1)\) satisfy the assumptions made in a)? Compute \(\mu\) in this case.

Exercise 2. (the birthday problem)

Let \((X_n, n \geq 1)\) be a sequence of i.i.d. random variables, each uniform on \(\{1, \ldots, N\}\). Let also

\[
T_N = \min\{n \geq 1 : X_n = X_m \text{ for some } m < n\}
\]

(notice that whatever happens, \(T_N \in \{2, \ldots, N + 1\}\)). Show that

\[
\mathbb{P}\left(\frac{T_N}{\sqrt{N}} \leq t\right) \xrightarrow{N \to \infty} 1 - e^{-t^2/2}, \quad \forall t \geq 0
\]

Remarks:
- Approximations are allowed here!
- Please observe that the limit distribution is not the Gaussian distribution!

Numerical application: Use this to obtain a rough estimate of \(\mathbb{P}(\{T_{365} \leq 22\})\) and \(\mathbb{P}(\{T_{365} \leq 50\})\)
(i.e., what is the probability that among 22 / 50 people, at least two share the same birthday?)