APA quiz 12

1. Let \((X_n, n \geq 1)\) be a sequence of iid r.v.'s such that \(P(\exists X_i = +1) = P(\exists X_i = -1) = \frac{1}{2}\). Which of the following processes are square-integrable martingales?

a) \(\Pi_0 = 0, \quad \Pi_n = X_n \quad n \geq 1\)

b) \(\Pi_0 = 1, \quad \Pi_{n+1} = \Pi_n + X_{n+1}, \quad n \in \mathbb{N}\)

c) \(\Pi_0 = 1, \quad \Pi_{n+1} = \frac{\Pi_n}{2} + X_{n+1}, \quad n \in \mathbb{N}\)

d) \(\Pi_0 = 1, \quad \Pi_{n+1} = \Pi_n \cdot X_{n+1}, \quad n \in \mathbb{N}\)

e) \(\Pi_0 = 1, \quad \Pi_{n+1} = \Pi_n (1 + X_{n+1}), \quad n \in \mathbb{N}\)

f) \(\Pi_0 = 1, \quad \Pi_{n+1} = \Pi_n \left(1 + \frac{X_{n+1}}{2}\right), \quad n \in \mathbb{N}\)
2. Let $M$ be a martingale and $N$ be a submartingale. (With respect to the same filtration $(\mathcal{F}_n, n \in \mathbb{N})$)

Which of the following processes $X$ is necessarily also a submartingale? (putting aside integrability conditions)

a) $X = M + N$
b) $X = (M + N)^2$
c) $X = \exp(N)$
d) $X = |M|
eq$
e) $X = M - N$
f) $X = -\log(M)$ (assuming here $M_n > 0 \text{ the } N$)
3. Let \((X_n, n \in \mathbb{N})\) be a discrete-time stochastic process and \(\mathcal{F}_n = \sigma(X_0, \ldots, X_n)\) for \(n \in \mathbb{N}\). Which of the following random times \(T\) are also stopping times w.r.t. \((\mathcal{F}_n, n \in \mathbb{N})\)?

a) \(T = \inf \{ n \geq 1 : X_n > \max (X_0, \ldots, X_{n-1}) \}\)

b) \(T = \inf \{ n \geq 0 : |X_n| \leq a \}\) for some \(a > 0\)

c) \(T = \sup \{ n \geq 0 : |X_n| \geq a \}\)

d) \(T = \inf \{ n \geq 0 : X_n \leq X_{n+1} \}\)

e) \(T = \min (N, \inf \{ n \geq 0 : X_n \geq X_{n-1} \})\) for some \(N \in \mathbb{N}\)

f) \(T = \inf \{ n \geq 0 : X_n \geq X_{n+1} \forall n \in \mathbb{N} \}\)
4. Let \( M \) be a martingale w.r.t. a filtration \((\mathcal{F}_n, n \in \mathbb{N})\) and \( T \) be a stopping time w.r.t. \((\mathcal{F}_n, n \in \mathbb{N})\).

Which of the following statements are always correct?

\[ a) \ \mathbb{E}(M_T) = \mathbb{E}(M_0) \]

\[ b) \ \text{If } T(\omega) \leq N \ \forall \omega \in \Omega, \text{ then } \mathbb{E}(M_T^2) \leq \mathbb{E}(M_N^2) \]

\[ c) \ \text{Affirmation b) is true even if } T \text{ is not a stopping time.} \]

\[ d) \ \mathbb{E}(M_T) \geq \mathbb{E}(M_0) \]

\[ e) \ \mathbb{E}(|M_T|) \geq \mathbb{E}(|M_0|) \]

\[ f) \ \text{If } T(\omega) \leq N \ \forall \omega \in \Omega, \text{ then } \exists C > 0 \text{ such that } |M_{T(\omega)}| \leq C \ \forall \omega \in \Omega \]