Coloring Problem: Analysis of the Metropolis chain.

Recap the problem.

\[ G = (V, E) \]

Sample from the space of proper colorings.

i.e. set two neighboring vertices \((u, v) \in E\)
don't have same color.

Notation: \( \{1, 2, 3 \ldots q\} = \text{set of colors} \).

\[ \Delta = \max \text{ degree of a vertex } v \in V. \]

\[ x = (x_1, x_2, \ldots, x_N) \quad N = |V| \]

and \( x_N \) color assigned to \( N \in V \).

\[ \pi(x) = \frac{\Pi(x \text{ is proper})}{Z} \]

\( Z \) counts total # of proper cols.
Recap the proposed algorithm.

1. Start from an initial proper coloring.
2. Select \( N \in V \) uniformly at random.
3. Select a color \( c \in \{1, 2, \ldots, q\} \) uniformly at random.
4. Recolor vertex \( v \) iff \( c \) is an allowed color. (otherwise we do nothing.)

Recap the theorem.

**Theorem:** Assume \( q > 3\Delta \) then for any initial proper coloring \( \pi \),

\[
\| \mathcal{P}_x^m - \pi \|_{TV} \leq \frac{m}{N} \left(1 - \frac{3\Delta}{q}\right).
\]

dist with initial color \( x \) after \( m \) iterations.

and the mixing time \( T_\varepsilon = \inf \{ m \geq 1 : \max_{x \text{ proper}} \| \mathcal{P}_x^m - \pi \|_{TV} \leq \varepsilon \} \)

satisfies

\[
T_\varepsilon \leq \left(1 - \frac{3\Delta}{q}\right)^{-1} \left\{ N \log N + \log \frac{1}{\varepsilon} \right\}.
\]
Remark: The theorem with a similar but more advanced proof holds for $q > 2\Delta$.

$q$ smaller: out of our scope here and the problem becomes much harder.

($q$ too small; then breaks down.)

Remark:

1. $q = \Delta + 1 \Rightarrow$ certainly we can color the graph.

But the chain of the edge might not be irreducible. For example:

$\Delta = 2$ and $q = 3$

Can you move out of this configuration by following the edge steps?
It is a fact that for $q > 3\Delta$ the chain is irreducible.

But also for $q > \Delta + 2$ as one can show.

follow up this during the quiz session.

for a little proof.

Remark: The proof of the theorem will use this fact that for $q > 3\Delta$ the chain is irreducible.

The proof will proceed by a coupling argument:

$X_m$ and $Y_m$ - coupled chains.

Recover the property: $\| P_x^m - Q^m \|_V = \inf \mathbb{P}(X_m \neq Y_m)$ [couplings]
Proof of Theorem.

**Coupled chains.**

1. \((X_m, m \geq 0)\) the chain starting at \(X_0 = x\) with proper initial coloring, and follows the steps of

   - Select \(n \in V\) at random, select \(c \in (1, \ldots, p)\) at random, recolor if \(c\) is allowed.

2. \((Y_m, m \geq 0)\) starts at \(Y_0 \sim U\) and follows the same steps as \((X_m, m \geq 0)\) with

   - The same \(n \in V\) and \(c \in (1, \ldots, p)\) at each time step.

   So the chains are coupled.

**Hamming distance between chains at each time step.**

\[
d(X_m, Y_m) = \sum_{n \in V} \mathbb{1}(x_n^{(m)} \neq y_n^{(m)})
\]
Since this is some coupling between chain:

\[ \| \bar{P}_x^n - \bar{\pi} \|_TV \leq \mathbb{P}(X_m \neq Y_m) \]

at all times

dist of chain \((Y_m, m \geq 0)\)

is \(\bar{\pi}\) because \(\bar{\pi}\)
is sll dist.

Markov inequality:

\[ \mathbb{E}(d(X_m, Y_m)) \]

We have a new inequality to start with:

\[ \| \bar{P}_x^n - \bar{\pi} \|_TV \leq \mathbb{E}(d(X_m, Y_m)) \]

\[
\leq N \exp \left( -\frac{m}{N} \left( 1 - \frac{S \Delta}{\rho} \right) \right)
\]

we are going to bound this expectation now, to prove
Proceed by induction:

• First we assume that $d(X_0, Y_0) = 1$

prove the bound.

• Then we generalize to $d(X_0, Y_0) = r \geq 1$.

• Finally we conclude.

Assume at some $m = 0$, $d(X_0, Y_0) = 1$.

Then we assume $\exists N \in \mathbb{N}$ s.t. $x_n \neq y_n$ (at some $N$)

and $x_w = y_w \implies w = n$.

$\Rightarrow d(X_1, Y_1) \in \{0, 1, 2\}$.

$\Rightarrow \mathbb{E}(d(X_1, Y_1)) = 0 \cdot \mathbb{P}(d(X_1, Y_1) = 0) + 1 \cdot \mathbb{P}(d(X_1, Y_1) = 1) + 2 \cdot \mathbb{P}(d(X_1, Y_1) = 2) = 1 - \mathbb{P}(d(X_1, Y_1) = 0) + \mathbb{P}(d(X_1, Y_1) = 2)$.
If \( d(x_0, t_0) = 1 \),

\[
\mathbb{E}(d(x, t_1)) = 1 - \mathbb{P}(d(x, t_1) = 0) + \mathbb{P}(d(x, t_1) = 2).
\]

\[
\mathbb{P}(d(x, t_1) = 0) = \frac{1}{N} \frac{\text{# allowed colors}}{q} \geq \frac{1}{N} \frac{9-\Delta}{q}.
\]

\[
\mathbb{P}(d(x, t_1) = 2) \leq \frac{\Delta}{N} \frac{2}{q}.
\]

- If the selected vertex \( w \) is not a neighbor of \( v \), you will do the same recoloring or non-recoloring of \( w \) in both chains and \( d(x, t_1) \) remains 1.

- The event \( d(x, t_1) = 2 \) happens only if \( w \) is a neighbor. It should be that you recolor \( w \) in both \( X \) and \( Y \) or recolor \( w \) in \( X \) and do nothing in \( Y \) or do nothing in both. 

The graphs illustrate the relationship between vertices and edges, highlighting the concept of recoloring and its implications on the graph's structure.
\[ E(d(x, y)) \leq 1 - \frac{1}{N} (1 - \frac{32}{q}) \]

Generalize to case \( d(x_0, f_0) = 1 \).

Claim by irreducibility (for \( q > 3 \)) or in fact for \( q > 3 + 2 \).

A path between assignments:

\[ X_0 \to Z_0 \to Z_1 \to \ldots \to Z_k \to \cdots \to Z_0 \to X_0. \]

Set \( d(Z_0, Z_0) = 1 \); all dist between assignments are equal to 1.

Let evolve the chains by one time unit:

\[ X_1 \to Z_1 \to Z_1 \to \ldots \to Z_k \to \cdots \to Z_1 \to X_1. \]

By triangle inequality:

\[ d(x, y) \leq d(x, Z_0^{(0)}) + d(Z_0^{(0)}, Z_1^{(0)}) + \ldots + d(Z_k^{(k)}, Z_1^{(1)}). \]

Take the expectation, use linearity, use result (\( \mathbb{E} \)) under \( \mathcal{V} \)

\[ \mathbb{E}(d(x, y)) \leq 1 - \frac{1}{N} (1 - \frac{32}{q}) \]
Conclusion of proof:

Remark by homogeneity of the Markov chain:

$$\mathbb{E}(d(X_{m+1}, Y_{m+1}) \mid d(X_m, Y_m) = r)$$

$$\leq r \left( 1 - \frac{1}{N} (1 - \frac{3d}{9}) \right)$$

Average over $r$:

$$\sum_{r} \mathbb{E}(d(X_{m+1}, Y_{m+1}) \mid d(X_m, Y_m) = r) \mathbb{P}(d(X_m, Y_m) = r)$$

$$= \mathbb{E}(d(X_{m+1}, Y_{m+1}))$$

$$\leq \left( 1 - \frac{1}{N} (1 - \frac{3d}{9}) \right) \sum_{r} r \mathbb{P}(d(X_m, Y_m) = r) \mathbb{E}(d(X_m, Y_m))$$

$$= D.$$
We found:

\[ E(d(x_{m_0}, y_{m_0})) \leq E(d(x_0, t)) \left\{ 1 - \frac{1}{N} (1 - \frac{32}{9}) \right\}. \]

\[ \leq \frac{E(d(x_0, t))}{\frac{1}{N} (1 - \frac{32}{9})} \]

\[ \leq N \]

Total # of vertex.

\[ 2^D \leq E(d(x_{m_0}, y_{m_0})) \leq N \left( 1 - \frac{1}{N} (1 - \frac{32}{9}) \right)^N. \]

Use \( 1 - x \leq e^{-x} \)

QED. \( \Box \)