Solutions 4

1. a) $\mathbf{1} \leq \mathbf{2}$: Consider $A = \{j \in S : \mu_j \geq \nu_j\}$. Then,

$$\begin{aligned}\sum_{j\in S} |\mu_j - \nu_j| &= \sum_{j\in A}(\mu_j - \nu_j) + \sum_{j\in A^c}(\nu_j - \mu_j) = \sum_{j\in A}(\mu_j - \nu_j) + \sum_{j\in S}(\nu_j - \mu_j) - \sum_{j\in A}(\nu_j - \mu_j) \\ &= 2\left(\sum_{j\in A}(\mu_j - \nu_j)\right) = 2(\mu(A) - \nu(A))\end{aligned}$$

as $\sum_{j\in S} \mu_j = \sum_{j\in S} \nu_j = 1$. Therefore,

$$\sup_{A\subset S} |\mu(A) - \nu(A)| \geq \frac{1}{2}\sum_{j\in S} |\mu_j - \nu_j|$$

$\mathbf{2} \leq \mathbf{3}$: For every $A \subset S$, define

$$\phi_j = \begin{cases} +1 & \text{if } j \in A \\ -1 & \text{if } j \in A^c \end{cases}$$

Then

$$\mu(\phi) - \nu(\phi) = \sum_{j\in A}(\mu_j - \nu_j) - \sum_{j\in A^c}(\mu_j - \nu_j) = 2\,(\mu(A) - \nu(A))$$

by the same argument as above. Hence,

$$\frac{1}{2}\sup_{\phi:S\to[-1,+1]} |\mu(\phi) - \nu(\phi)| \geq \sup_{A\subset S} |\mu(A) - \nu(A)|$$

$\mathbf{3} \leq \mathbf{1}$: Simply observe that for every $\phi : S \to [-1, +1]$, we have

$$|\mu(\phi) - \nu(\phi)| = \left|\sum_{j\in S}(\mu_j - \nu_j)\,\phi_j\right| \leq \sum_{j\in S} |\mu_j - \nu_j|\,|\phi_j| \leq \sum_{j\in S} |\mu_j - \nu_j|$$

Therefore,

$$\frac{1}{2}\sup_{\phi:S\to[-1,+1]} |\mu(\phi) - \nu(\phi)| \leq \frac{1}{2}\sum_{j\in S} |\mu_j - \nu_j|$$

b1) $\mathbb{P}(X = Y = 1) = \mathbb{P}(X = Y = 0) = 1/4$, $\mathbb{P}(X = 0, Y = 1) = 1/2$ and $\mathbb{P}(X = 1, Y = 0) = 0$. In this case, $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = 1/4 - (1/4)\,(3/4) = 1/16 > 0$.

b2) $\mathbb{P}(X = Y = 1) = \mathbb{P}(X = Y = 0) = 3/16$, $\mathbb{P}(X = 0, Y = 1) = 9/16$ and $\mathbb{P}(X = 1, Y = 0) = 1/16$. In this case $\mathbb{P}(X = i, Y = j) = \mathbb{P}(X = i)\,\mathbb{P}(Y = j)$, so X and Y are independent, which implies that $\mathrm{Cov}(X, Y) = 0$.

b3) $\mathbb{P}(X = Y = 1) = \mathbb{P}(X = Y = 0) = 0$, $\mathbb{P}(X = 0, Y = 1) = 3/4$ and $\mathbb{P}(X = 1, Y = 0) = 1/4$. In this case, $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = 0 - (1/4)\,(3/4) = -3/16 < 0$.

c) Observe that $||\mu - \nu||_{\mathrm{TV}} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$ and that the coupling in **b1**) satisfies $\mathbb{P}(X \neq Y) = 1/2$. Note also that $\mathbb{P}(X \neq Y) > 1/2$ for the other two (as shown in the course).

2. a) With some basic calculations, we obtain $\pi_0 = \frac{q}{p+q}$ and $\pi_1 = \frac{p}{p+q}$.

b) The transition matrix over the state space $\mathcal{S} \times \mathcal{S} = \{(0,0), (0,1), (1,0), (1,1)\}$ is given by

$$P_Z = \begin{pmatrix} 1-p & 0 & 0 & p \\ (1-p)q & (1-p)(1-q) & pq & p(1-q) \\ (1-p)q & pq & (1-p)(1-q) & p(1-q) \\ q & 0 & 0 & 1-q \end{pmatrix}$$

c) States $(0,1)$ and $(1,0)$ are transient, and states $(0,0)$ and $(1,1)$ are recurrent.

d) Z admits a unique stationary distribution, which is zero over the transient states and is equal to π for the recurrent states. Formally, the stationary distribution of Z is

$$\pi_Z = \left(\frac{q}{p+q}, 0, 0, \frac{p}{p+q} \right)$$

e) Let us define the event $A_n = \{X_n \neq Y_n\}$. As $A_{n+1} \subset A_n$ by definition of the coupling, we obtain

$$\begin{aligned} &\mathbb{P}(A_{n+1}) = \mathbb{P}(A_{n+1} \cap A_n) = \mathbb{P}(A_{n+1} \cap \{Z_n = (0,1)\}) + \mathbb{P}(A_{n+1} \cap \{Z_n = (1,0)\}) \\ &= \mathbb{P}(A_{n+1} | \{Z_n = (0,1)\}) \, \mathbb{P}(\{Z_n = (0,1)\}) + \mathbb{P}(A_{n+1} | \{Z_n = (1,0)\}) \, \mathbb{P}(\{Z_n = (1,0)\}) \\ &= (pq + (1-p)(1-q)) \, (\mathbb{P}(Z_n = (0,1)) + \mathbb{P}(Z_n = (1,0))) = (pq + (1-p)(1-q)) \, \mathbb{P}(A_n) \end{aligned}$$

where we used P_Z from part b). Therefore,

$$\mathbb{P}(X_{n+1} \neq Y_{n+1}) = \Big(pq + (1-p)(1-q) \Big) \, \mathbb{P}(X_n \neq Y_n)$$

f) Based on the material of the course, if X_0 has an initial distribution $\pi^{(0)}$ and Y_0 has the stationary distribution π as its initial distribution, we obtain

$$\|\pi^{(n)} - \pi\|_{\text{TV}} \leq \mathbb{P}(X_n \neq Y_n) = \Big(pq + (1-p)(1-q) \Big) \, \mathbb{P}(X_{n-1} \neq Y_{n-1})$$

which naturally leads to

$$\|\pi^{(n)} - \pi\|_{\text{TV}} \leq \Big(pq + (1-p)(1-q) \Big)^n \, \mathbb{P}(X_0 \neq Y_0)$$

Then, considering $\pi^{(0)}$ as a distribution concentrated on a single state, we obtain

$$\max_{i \in \mathcal{S}} \|P_i^n - \pi\|_{TV} \leq \Big(pq + (1-p)(1-q) \Big)^n \max \left\{ \frac{q}{p+q}, \frac{p}{p+q} \right\} \leq \Big(pq + (1-p)(1-q) \Big)^n$$

g) Let us define the function $f(p) = p^2 + (1-p)^2$. Then, for the case that $p = q$, the upper bound found in the previous part can be written as

$$\max_{i \in \mathcal{S}} \|P_i^n - \pi\|_{\text{TV}} \leq f(p)^n$$

Hence, the fastest convergence corresponds to the minimum value of $f(p)$, which can be found by solving

$$f'(p) = 2p - 2(1-p) = 0$$

whose solution is $p^* = \frac{1}{2}$.

3. a) The transition matrix is given by

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

b) The chain is irreducible since the transition graph is strongly connected. So whatever state we are in, there is a path with non-zero probability to any other state.

The chain is periodic since starting from any state, we can return to it only using an even number of steps. Since this number can be as small as 2 for each state, we find that their period is 2.

Since the chain is not aperiodic, it is not ergodic.

c) Solving the system of linear equations $\pi = \pi P$, we find that

$$\pi = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right).$$

Note that since the chain is not ergodic, π is not a limiting distribution.

d) We want to compute the mean recurrence time $\mu_2 = \mathbb{E}(T_2 | X_0 = 2)$, which by the remark under theorem 1.2 in the lecture notes is given in this case by $\mu_2 = \frac{1}{\pi_2} = 3$ since the chain is irreducible and has stationary distribution π.

e) This can be computed using the Chapman-Kolmogorov equations, which amounts to computing P^2. We find that

$$P^2 = \begin{pmatrix} 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 1/8 & 1/8 & 0 & 3/8 & 3/8 & 0 \\ 1/8 & 1/8 & 0 & 3/8 & 3/8 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

For any $i, j \in \mathcal{S}$, we have that $p_{ij}^{(2)} = \left(P^2\right)_{ij}$.

f) From the previous answer, we easily see from the transition matrix $Q = P^2$ that there are two equivalence classes. The first one comprises states $\{0, 1, 3, 4\}$ and the other class comprises states $\{2, 5\}$. There is no transition possible between these two classes.

g) Recall from part c) that there is no limiting distribution for $(X_n, n \geq 0)$. So the limit $\lim_{n\to\infty} \mathbb{P}(X_n = 1 | X_0 = 1)$ does not exist.

Now for the second limit, we need to consider the transition matrix Q. Note that starting from state 1, we will only stay in states that are in the equivalence class $\{0, 1, 3, 4\}$ found in part f). So the question is, does this reduced chain has a limiting distribution? The transition matrix for that equivalence class is

$$Q_{\{0,1,3,4\}} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/8 & 1/8 & 3/8 & 3/8 \\ 1/8 & 1/8 & 3/8 & 3/8 \end{pmatrix}.$$

With this, we see that the reduced chain is finite, irreducible and aperiodic (since there are self-loops), and thus it is ergodic and has a stationary and limiting distribution that we will call ν. Solving $\nu = \nu Q_{\{0,1,3,4\}}$, we find that

$$\nu = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right).$$

Thus, we get $\lim_{n\to\infty} \mathbb{P}(X_{2n} = 1 | X_0 = 1) = \nu_1 = \frac{1}{6}$.