Which of the following chains have a small spectral gap? Why?
(assuming unmarked arrows have balanced weights)

(a) \[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

(b) \[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

\[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

(b1) \( p = q = 0.01 \)

(b2) \( p = q = 0.99 \)

c) \[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

\[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

d) \[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

\[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6
\end{array} \]

(d1) \( p = q = 0.01 \)

(d2) \( p = q = 0.99 \)
2) For which of the previous chains having a small spectral gap does the addition of self-loops (all of the same weight) increase the spectral gap (and therefore the convergence rate)?

3) Let \((D_j, j \in S)\) be probabilities s.t. \(p_j > 0 \ \forall j \in S \ \& \ \sum_{j \in S} p_j = 1\).

a) Show that if \(\exists j \in S\) with \(o > 0 \ \forall j \in S \ \& \ \sum_{j \in S} p_j c_j = c_i\), then \(c_j = c_i \ \forall j \in S\).

b) Show that if \(d_j \in \{ \pm 1 \} \ \forall j \in S\) and \(\sum_{j \in S} p_j d_j = d_i\) for some \(i \in S\), then \(d_j = d_i \ \forall j \in S\).

c) Show that \(\exists d \in \{ \pm 1 \}^S\) s.t. \(\sum_{j \in S} p_j d_j = -d_i\) for some \(i \in S\).