Solutions 2

1. a) The process Y is a Markov chain. Here is the proof:

$$\begin{aligned}&\mathbb{P}(Y_{n+1} = j | Y_n = i, Y_{n-1} = i_{n-1}, \ldots, Y_1 = i_1, Y_0 = 0)\\&= \mathbb{P}(S_{2n+2} = j | S_{2n} = i, S_{2n-2} = i_{n-1}, \ldots, S_2 = i_1, S_0 = 0)\\&= \mathbb{P}(S_{2n} + X_{2n+1} + X_{2n+2} = j | S_{2n} = i, S_{2n-2} = i_{n-1}, \ldots, S_2 = i_1, S_0 = 0)\\&= \mathbb{P}(X_{2n+1} + X_{2n+2} = j - i)\end{aligned}$$

and also

$$\begin{aligned}&\mathbb{P}(Y_{n+1} = j | Y_n = i) = \mathbb{P}(S_{2n+2} = j | S_{2n} = i) = \mathbb{P}(S_{2n} + X_{2n+1} + X_{2n+2} = j | S_{2n} = i)\\&= \mathbb{P}(X_{2n+1} + X_{2n+2} = j - i)\end{aligned}$$

so the process Y is a Markov chain. Moreover,

$$\mathbb{P}(X_{2n+1} + X_{2n+2} = j - i) = \begin{cases} 1/4, & \text{if } |j - i| = 2, \\ 1/2, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

These probabilities do not depend on n, so the process Y is a time-homogeneous Markov chain.

b) The process Z is a Markov chain. Actually, for $n \geq 0$, we have $Z_{n+1} = (-1)^{S_n + X_{n+1}} = (-1)^{S_n} (-1)^{X_{n+1}} = -Z_n$ always, as $(-1)^{X_{n+1}} = -1$, irrespective of the value of $X_{n+1} \in \{-1, +1\}$. The process Z is therefore deterministic, constantly alternating between the two states $+1$ and -1, and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c) The process T is not a Markov chain. Here is why: on the one hand, we have (by counting the number of possible paths):

$$\mathbb{P}(T_4 = 1 | T_3 = 1, T_2 = 1) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 1)}{\mathbb{P}(T_3 = 1, T_2 = 1)} = \frac{3/16}{2/8} = \frac{3}{4}$$

On the other hand, we have:

$$\mathbb{P}(T_4 = 1 | T_3 = 1, T_2 = 0) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 0)}{\mathbb{P}(T_3 = 1, T_2 = 0)} = \frac{1/16}{1/8} = \frac{1}{2} \neq \frac{3}{4}$$

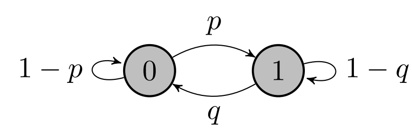
so the process T is not a Markov chain.

d) The process W is exactly the same as the process Y, so it is a Markov chain.

2. a) Transition matrix:

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

Transition graph:



b) X_{n+1} is independent of X_n if and only if for all $(i,j) \in \mathcal{S}^2$, it holds that $\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j)$. Therefore, we have the necessary condition $\mathbb{P}(X_{n+1} = 1 | X_n = 1) = \mathbb{P}(X_{n+1} = 1 | X_n = 0)$, that is: $1 - q = p$, and thus $p + q = 1$.

Conversely, the transition matrix $P = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$ satisfies the aforementioned independence condition. Note that the time-homogeneity condition of the Markov chain extends the independence.

c) P has the characteristic polynomial $\chi_P(X) = ((1-p) - X)((1-q) - X) - pq$, that is:

$$\chi_P(X) = X^2 - (2 - p - q)X + 1 - p - q$$

whose discriminant is positive since $p, q > 0$ and:

$$\Delta = (1 + 1 - p - q)^2 - 4(1 - p - q) = (p+q)^2$$

Thus there are two distinct eigenvalues: $\{1, 1-p-q\}$, with respective eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} p \\ -q \end{pmatrix}$.

Writing $D = \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}$ and $U = \begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix}$ we have:

$$P = UDU^{-1}$$

where:

$$U^{-1} = \frac{1}{p+q} \begin{pmatrix} q & p \\ 1 & -1 \end{pmatrix}$$

Thus for $n \in \mathbb{N}$:

$$\boxed{p_{0,0}^{(n)} = [P^n]_{0,0} = [UD^nU^{-1}]_{0,0} = \frac{q + p(1-p-q)^n}{p+q}}$$

d) We have for $N \in \mathbb{N}^*$:

$$\sum_{n=1}^{N} p_{00}^{(n)} = \frac{1}{p+q} \left(qN + p(1-p-q) \frac{1 - (1-p-q)^{N-1}}{p+q} \right)$$

Because $0 < p < 1, 0 < q < 1$, we have $|1 - p - q| < 1$ and thus the second term converges when $N \to \infty$. As for the first term, it converges only when $q = 0$, which is not the case by assumption. Therefore:

$$\sum_{n \geq 1} p_{00}^{(n)} = +\infty$$

and we conclude that state 0 is recurrent.

e) We have:

$$\begin{aligned}
f_{00}^{(n)} &= \mathbb{P}(X_n = 0, X_{n-1} = \cdots = X_1 = 1 | X_0 = 0) \\
&= \mathbb{P}(X_n = 0, X_{n-1} = \cdots = X_2 = 1 | X_1 = 1, X_0 = 0) P(X_1 = 1 | X_0 = 0) \\
&= \mathbb{P}(X_n = 0, X_{n-1} = \cdots = X_2 = 1 | X_1 = 1) P(X_1 = 1 | X_0 = 0) \\
&= \ldots \\
&= \mathbb{P}(X_n = 0 | X_{n-1} = 1) P(X_{n-1} = 1 | X_{n-2} = 1) \cdots P(X_1 = 1 | X_0 = 0) \\
&= \begin{cases} p_{1,0} \cdot p_{1,1}^{n-2} \cdot p_{0,1} & (n > 1) \\ p_{0,0} & (n = 1) \end{cases}
\end{aligned}$$

thus:

$$\boxed{f_{00}^{(n)} = \begin{cases} pq\,(1-q)^{n-2} & (n > 1) \\ 1 - p & (n = 1) \end{cases}}$$

Then:

$$f_{00} = \sum_{n \geq 1} f_{00}^{(n)} = pq\,\frac{1}{1 - (1 - q)} + (1 - p) = 1$$

We thus find again that state 0 is recurrent.

f) We have:

$$\mu_0 = \mathbb{E}\,(T_0 | X_0 = 0) = \sum_{n \geq 1} n \mathbb{P}(T_0 = n | X_0 = 0) = f_{00}^{(1)} + \sum_{n \geq 2} n f_{00}^{(n)}$$

Notice the following power series relationship for $x \in (-1, 1)$:

$$\sum_{n \geq 0} x^n = \frac{1}{1-x} \Longrightarrow x \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n \geq 0} x^n = x \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} \Longrightarrow \sum_{n \geq 0} n x^n = \frac{x}{(1-x)^2}$$

and rewrite the former expression as:

$$\begin{aligned}
\mu_0 = \mathbb{E}\,(T_0 | X_0 = 0) &= (1 - p) + pq \sum_{n \geq 2} (n - 2 + 2)(1 - q)^{n-2} \\
&= (1 - p) + pq \sum_{n \geq 0} n(1 - q)^n + 2pq \sum_{n \geq 0} (1 - q)^n \\
&= 1 - p + pq\,\frac{1 - q}{q^2} + 2pq\,\frac{1}{q}
\end{aligned}$$

in conclusion:

$$\boxed{\mu_0 = \mathbb{E}\,(T_0 | X_0 = 0) = 1 + \frac{p}{q}}$$

This is finite since $q > 0$, so state 0 is positive-recurrrent.

g) Considering the two cases:

1. when $p+q=1$, we have $f_{00}^{(n)} = q\,(1-q)^{n-1}$ for $n \geq 1$, so T_0 is a geometric random variable with parameter q (conditioned on the fact that $X_0 = 0$). Correspondingly, $\mu_0 = \mathbb{E}\,(T_0|X_0 = 0) = \frac{1}{q}$. So if q is close to 0, then p is close to 1 and there is a higher chance to jump to state 1 and lower chance to go back to state 0. We see that the mean μ_0 is getting higher in such case. Conversely, when q is close to 1, p is close to 0 and the process tends to stick to state 0.

2. when $p = q$ we simply have $\mu_0 = 2$. Indeed: even if, for instance, p is close to 0 a very unlikely jump to state 1 means to get stuck in 1 for a proportional "higher" period - all in all, we get a mean of 2 steps.

3. a) Let i be a recurrent state and j be another state in the same equivalence class. i and j communicate, so there exist $n_1, n_2 \geq 1$ such that $p_{ji}^{(n_1)} > 0$ and $p_{ij}^{(n_2)} > 0$. As i is recurrent, we know that

$$\sum_{n\geq 1} p_{ii}^{(n)} = +\infty$$

Besides, for every $n \geq 1$, we have $p_{jj}^{(n_1+n+n_2)} \geq p_{ji}^{(n_1)}\,p_{ii}^{(n)}\,p_{ij}^{(n_2)}$, so because of the assumptions made:

$$\sum_{n\geq 1} p_{jj}^{(n_1+n+n_2)} \geq p_{ji}^{(n_1)} \left(\sum_{n\geq 1} p_{ii}^{(n)} \right) p_{ij}^{(n_2)} = +\infty$$

and we therefore also have $\sum_{n\geq 1} p_{jj}^{(n)} = +\infty$, i.e., j is recurrent.

b) We imitate the proof given in the lectures. Let $A_m = \{X_m = j\}$ and $B_m = \{X_m = j, X_r \neq j,$ for $1 \leq r < m\}$. The events B_m are disjoint, so

$$\begin{aligned}
\mathbb{P}(A_m|X_0 = i) &= \sum_{r=1}^{m} \mathbb{P}\left(A_m \bigcap B_r \,\middle|\, X_0 = i\right) \\
&= \sum_{r=1}^{m} \mathbb{P}(A_m|B_r, X_0 = i)\,\mathbb{P}(B_r|X_0 = i) = \sum_{r=1}^{m} \mathbb{P}(A_m|X_r = j)\,\mathbb{P}(B_r|X_0 = i)
\end{aligned}$$

where we have used the Markov condition in the last equality. Hence

$$p_{ij}^{(m)} = \sum_{r=1}^{m} p_{jj}^{(m-r)}\,f_{ij}^{(r)}$$

Multiplying by s^m, $|s| < 1$ and summing over m, we find

$$P_{ij}(s) = P_{jj}(s)\,F_{ij}(s)$$

which is the desired result.

c)

1. If j is recurrent, we have $f_{jj} = 1$ by definition and we know from $P_{jj}(s) = 1+P_{jj}(s)F_{jj}(s)$ that $P_{jj}(1) = +\infty$. Then $P_{ij}(1) = +\infty$ as long as $F_{ij}(1) > 0$. This means that $\sum_{n\geq 0} p_{ij}^{(n)} = +\infty$ for i s.t. $f_{ij} > 0$ (use Abel's theorem like in class to take $\lim_{s\uparrow 1}$).

2. If j is transient, $P_{jj}(1) < +\infty$ (because $f_{jj} < 1$) and since $F_{ij}(1) \leq 1$, this means $P_{ij}(1) < +\infty$ which in turn means $\sum_{n\geq 0} p_{ij}^{(n)} < +\infty$.