1. a) The process $Y$ is a Markov chain. Here is the proof:

$$\mathbb{P}(Y_{n+1} = j|Y_n = i, Y_{n-1} = i_{n-1}, \ldots, Y_1 = i_1, Y_0 = 0)$$

$$= \mathbb{P}(S_{2n+2} = j|S_{2n} = i, S_{2n-2} = i_{n-1}, \ldots, S_2 = i_1, S_0 = 0)$$

$$= \mathbb{P}(S_{2n} + X_{2n+1} + X_{2n+2} = j|S_{2n} = i, S_{2n-2} = i_{n-1}, \ldots, S_2 = i_1, S_0 = 0)$$

$$= \mathbb{P}(X_{2n+1} + X_{2n+2} = j - i)$$

and also

$$\mathbb{P}(Y_{n+1} = j|Y_n = i) = \mathbb{P}(S_{2n+2} = j|S_{2n} = i) = \mathbb{P}(S_{2n} + X_{2n+1} + X_{2n+2} = j|S_{2n} = i)$$

$$= \mathbb{P}(X_{2n+1} + X_{2n+2} = j - i)$$

so the process $Y$ is a Markov chain. Moreover,

$$\mathbb{P}(X_{2n+1} + X_{2n+2} = j - i) = \begin{cases} 1/4, & \text{if } |j - i| = 2, \\ 1/2, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

These probabilities do not depend on $n$, so the process $Y$ is a time-homogeneous Markov chain.

b) The process $Z$ is a Markov chain. Actually, for $n \geq 0$, we have $Z_{n+1} = (-1)^{S_n + X_{n+1}} = (-1)^{S_n}(-1)^{X_{n+1}} = -Z_n$ always, as $(-1)^{X_{n+1}} = -1$, irrespective of the value of $X_{n+1} \in \{-1, +1\}$. The process $Z$ is therefore deterministic, constantly alternating between the two states $+1$ and $-1$, and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

c) The process $T$ is not a Markov chain. Here is why: on the one hand, we have (by counting the number of possible paths):

$$\mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 1) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 1)}{\mathbb{P}(T_3 = 1, T_2 = 1)} = \frac{3/16}{2/8} = \frac{3}{4}$$

On the other hand, we have:

$$\mathbb{P}(T_4 = 1|T_3 = 1, T_2 = 0) = \frac{\mathbb{P}(T_4 = 1, T_3 = 1, T_2 = 0)}{\mathbb{P}(T_3 = 1, T_2 = 0)} = \frac{1/16}{1/8} = \frac{1}{2} \neq \frac{3}{4}$$

so the process $T$ is not a Markov chain.

d) The process $W$ is exactly the same as the process $Y$, so it is a Markov chain.
2. a) Transition matrix:

\[ P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix} \]

Transition graph:

\[ \begin{array}{c}
0 \quad \xrightarrow{1 - p} \quad 1 \\
1 \quad \xrightarrow{p} \quad 0
\end{array} \]

b) \( X_{n+1} \) is independent of \( X_n \) if and only if for all \((i, j) \in S^2\), it holds that \( P(X_{n+1} = j | X_n = i) = P(X_{n+1} = j) \). Therefore, we have the necessary condition \( P(X_{n+1} = 1 | X_n = 1) = P(X_{n+1} = 1 | X_n = 0) \), that is: \( 1 - q = p \), and thus \( p + q = 1 \).

Conversely, the transition matrix \( P = \begin{pmatrix} 1 - p & p \\ 1 - p & p \end{pmatrix} \) satisfies the aforementioned independence condition. Note that the time-homogeneity condition of the Markov chain extends the independence.

c) \( P \) has the characteristic polynomial \( \chi_P(X) = ((1 - p) - X)((1 - q) - X) - pq \), that is:

\[ \chi_P(X) = X^2 - (2 - p - q)X + 1 - p - q \]

whose discriminant is positive since \( p, q > 0 \) and:

\[ \Delta = (1 + 1 - p - q)^2 - 4(1 - p - q) = (p + q)^2 \]

Thus there are two distinct eigenvalues: \( \{1, 1 - p - q\} \), with respective eigenvectors \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} p \\ -q \end{pmatrix} \).

Writing \( D = \begin{pmatrix} 1 & 0 \\ 0 & 1 - p - q \end{pmatrix} \) and \( U = \begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix} \) we have:

\[ P = UD U^{-1} \]

where:

\[ U^{-1} = \frac{1}{p + q} \begin{pmatrix} q & p \\ 1 & -1 \end{pmatrix} \]

Thus for \( n \in \mathbb{N} \):

\[ P_{0,0}^{(n)} = [P^n]_{0,0} = [UD^n U^{-1}]_{0,0} = \frac{q + p(1 - p - q)^n}{p + q} \]

D) We have for \( N \in \mathbb{N}^* \):

\[ \sum_{n=1}^{N} P_{0,0}^{(n)} = \frac{1}{p + q} \left( qN + p(1 - p - q) \frac{1 - (1 - p - q)^{N-1}}{p + q} \right) \]

Because \( 0 < p < 1, 0 < q < 1 \), we have \( |1 - p - q| < 1 \) and thus the second term converges when \( N \to \infty \). As for the first term, it converges only when \( q = 0 \), which is not the case by assumption. Therefore:

\[ \sum_{n \geq 1} P_{0,0}^{(n)} = +\infty \]

and we conclude that state 0 is recurrent.
e) We have:

\[ f_{00}^{(n)} = P(X_n = 0, X_{n-1} = \cdots = X_1 = 1 | X_0 = 0) \]
\[ = P(X_n = 0, X_{n-1} = \cdots = X_2 = 1 | X_1 = 1, X_0 = 0) P(X_1 = 1 | X_0 = 0) \]
\[ = P(X_n = 0, X_{n-1} = \cdots = X_2 = 1 | X_1 = 1) P(X_1 = 1 | X_0 = 0) \]
\[ = \cdots \]
\[ = P(X_n = 0 | X_{n-1} = 1) P(X_{n-2} = 1 | X_{n-1} = 1) \cdots P(X_1 = 1 | X_0 = 0) \]

\[ = \begin{cases} p_{1,0} \cdot p_{1,1}^n \cdot p_{0,1} & (n > 1) \\ p_{0,0} & (n = 1) \end{cases} \]

Thus:

\[ f_{00}^{(n)} = \begin{cases} pq (1 - q)^{n-2} & (n > 1) \\ 1 - p & (n = 1) \end{cases} \]

Then:

\[ f_{00} = \sum_{n \geq 1} f_{00}^{(n)} = pq \frac{1}{1 - (1 - q)} + (1 - p) = 1 \]

We thus find again that state 0 is recurrent.

f) We have:

\[ \mu_0 = \mathbb{E}(T_0 | X_0 = 0) = \sum_{n \geq 1} n P(T_0 = n | X_0 = 0) = f_{00}^{(1)} + \sum_{n \geq 2} n f_{00}^{(n)} \]

Notice the following power series relationship for \( x \in (-1, 1) \):

\[ \sum_{n \geq 0} x^n = \frac{1}{1 - x} \implies x \frac{d}{dx} \sum_{n \geq 0} x^n = x \frac{d}{dx} \frac{1}{1 - x} \implies \sum_{n \geq 0} n x^n = \frac{x}{(1 - x)^2} \]

and rewrite the former expression as:

\[ \mu_0 = \mathbb{E}(T_0 | X_0 = 0) = (1 - p) + pq \sum_{n \geq 2} (n - 2 + 2)(1 - q)^{n-2} \]

\[ = (1 - p) + pq \sum_{n \geq 0} n(1 - q)^n + 2pq \sum_{n \geq 0} (1 - q)^n \]

\[ = 1 - p + pq \frac{1 - q}{q^2} + 2pq \frac{1}{q} \]

in conclusion:

\[ \mu_0 = \mathbb{E}(T_0 | X_0 = 0) = 1 + \frac{p}{q} \]

This is finite since \( q > 0 \), so state 0 is positive-recurrent.
g) Considering the two cases:

1. when \( p + q = 1 \), we have \( f^{(n)}_{00} = q(1-q)^{n-1} \) for \( n \geq 1 \), so \( T_0 \) is a geometric random variable with parameter \( q \) (conditioned on the fact that \( X_0 = 0 \)). Correspondingly, \( \mu_0 = \mathbb{E}(T_0|X_0 = 0) = \frac{1}{q} \). So if \( q \) is close to 0, then \( p \) is close to 1 and there is a higher chance to jump to state 1 and lower chance to go back to state 0. We see that the mean \( \mu_0 \) is getting higher in such case. Conversely, when \( q \) is close to 1, \( p \) is close to 0 and the process tends to stick to state 0.

2. when \( p = q \) we simply have \( \mu_0 = 2 \). Indeed: even if, for instance, \( p \) is close to 0 a very unlikely jump to state 1 means to get stuck in 1 for a proportional "higher" period - all in all, we get a mean of 2 steps.

3. a) Let \( i \) be a recurrent state and \( j \) be another state in the same equivalence class. \( i \) and \( j \) communicate, so there exist \( n_1, n_2 \geq 1 \) such that \( p^{(n_1)}_{ij} > 0 \) and \( p^{(n_2)}_{ij} > 0 \). As \( i \) is recurrent, we know that

\[
\sum_{n \geq 1} p^{(n)}_{ii} = +\infty
\]

Besides, for every \( n \geq 1 \), we have \( p^{(n_1+n+n_2)}_{jj} \geq p^{(n_1)}_{ji} p^{(n)}_{ii} p^{(n_2)}_{ij} \), so because of the assumptions made:

\[
\sum_{n \geq 1} p^{(n_1+n+n_2)}_{jj} \geq p^{(n_1)}_{ji} \left( \sum_{n \geq 1} p^{(n)}_{ii} \right) p^{(n_2)}_{ij} = +\infty
\]

and we therefore also have \( \sum_{n \geq 1} p^{(n)}_{jj} = +\infty \), i.e., \( j \) is recurrent.

b) We imitate the proof given in the lectures. Let \( A_m = \{ X_m = j \} \) and \( B_m = \{ X_m = j, X_r \neq j \text{ for } 1 \leq r < m \} \). The events \( B_m \) are disjoint, so

\[
\mathbb{P}(A_m|X_0 = i) = \sum_{r=1}^{m} \mathbb{P}(A_m \cap B_r | X_0 = i) = \sum_{r=1}^{m} \mathbb{P}(A_m|B_r, X_0 = i) \mathbb{P}(B_r|X_0 = i) = \sum_{r=1}^{m} \mathbb{P}(A_m|X_r = j) \mathbb{P}(B_r|X_0 = i)
\]

where we have used the Markov condition in the last equality. Hence

\[
P^{(m)}_{ij} = \sum_{r=1}^{m} P^{(m-r)}_{jj} f^{(r)}_{ij}
\]

Multiplying by \( s^m, |s| < 1 \) and summing over \( m \), we find

\[
P_{ij}(s) = P_{jj}(s) F_{ij}(s)
\]

which is the desired result.
c) 

1. If $j$ is recurrent, we have $f_{jj} = 1$ by definition and we know from $P_{jj}(s) = 1 + P_{jj}(s)F_{jj}(s)$ that $P_{jj}(1) = +\infty$. Then $P_{ij}(1) = +\infty$ as long as $F_{ij}(1) > 0$. This means that $\sum_{n \geq 0} p_{ij}^{(n)} = +\infty$ for $i$ s.t. $f_{ij} > 0$ (use Abel’s theorem like in class to take $\lim_{s \uparrow 1}$).

2. If $j$ is transient, $P_{jj}(1) < +\infty$ (because $f_{jj} < 1$) and since $F_{ij}(1) \leq 1$, this means $P_{ij}(1) < +\infty$ which in turn means $\sum_{n \geq 0} p_{ij}^{(n)} < +\infty$. 