Solutions 7

1. a) Let us write by convention that $y \sim x$ if there exists a unique $j \in \{1, \ldots, d\}$ such that $y_j \neq x_j$. Observing that the described process is a random walk on the graph described by the relation $\sim$, we deduce that the transition matrix of the chain is given by

$$p_{xy} = \begin{cases} \frac{1}{(m-1)d} & \text{if } y \sim x \\ 0 & \text{otherwise} \end{cases}$$

The chain is clearly irreducible, aperiodic and positive-recurrent, therefore ergodic. Its stationary distribution $\pi$ is uniform (i.e. $\pi_x = m^{-d}$ $\forall x \in S$), and the detailed balance equation is satisfied.

b) Assume that $|z| = k$ and denote by $A$ the set of indices $j \in \{1, \ldots, d\}$ such that $z_j \neq 0$ (so that $|A| = k$). Then

$$\left( P^{\phi(z)} \right)_x = \sum_{y \in S} p_{xy} \phi_y = \frac{1}{(m-1)d} \sum_{y \sim x} \exp \left( 2\pi i y \cdot z / m \right)$$

$$= \frac{1}{(m-1)d} \sum_{j=1}^d \sum_{t=0: t \neq x_j}^{m-1} \exp \left( 2\pi i \left( \sum_{l=1: l \neq j}^d x_l z_l + t z_j \right) / m \right)$$

$$= \frac{1}{(m-1)d} \sum_{j=1}^d \exp \left( 2\pi i \left( \sum_{l=1: l \neq j}^d x_l z_l \right) / m \right) \times \sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi i u z_j / m)$$

Observe now that if $z_j = 0$, then

$$\sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi i u z_j / m) = \sum_{u=0: u \neq x_j}^{m-1} 1 = m - 1 = (m - 1) \exp(2\pi i x_j z_j / m)$$

while if $z_j \neq 0$, then

$$\sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi i u z_j / m) = \sum_{u=0}^{m-1} \exp(2\pi i u z_j / m) - \exp(2\pi i x_j z_j / m) = 0 - \exp(2\pi i x_j z_j / m)$$

This finally gives

$$\left( P^{\phi(z)} \right)_x = \frac{1}{(m-1)d} \left( \sum_{j \in A} (-1) \exp(2\pi i x \cdot z / m) + \sum_{j \in A^c} (m-1) \exp(2\pi i x \cdot z / m) \right)$$

$$= \frac{1}{(m-1)d} \left( -d \exp(2\pi i x \cdot z / m) + (d - k) m \exp(2\pi i x \cdot z / m) \right) = \frac{(d - k) m - d}{(m-1)d} \phi_x^{(z)}$$

$$= \left( 1 - \frac{km}{(m-1)d} \right) \phi_x^{(z)}$$

The eigenvalue $\lambda_z$ corresponding to $\phi^{(z)}$ is therefore given by

$$\lambda_z = 1 - \frac{|z|m}{(m-1)d}$$
c) The second largest eigenvalue is equal to $1 - \frac{m}{(m-1)d}$, while the least eigenvalue is equal to $1 - \frac{1}{m-1}$. When $d > 2$ (remember also that by assumption, $m > 2$), the spectral gap is therefore determined by the second largest eigenvalue and equal to $\gamma = \frac{m}{(m-1)d}$. This leads to the following upper bound on the total variation distance:

$$\|P^n_0 - \pi\|_{TV} \leq \frac{1}{2\sqrt{\pi_0}} \exp(-\gamma n) = \frac{d}{2} \exp \left( -\frac{nm}{(m-1)d} \right),$$

which becomes small only when $n \geq cd^2 \log m$ for some constant $c > 0$.

d) The lower bound obtained in class applies here, as $|\phi_z^{(x)}|^2 = 1$ for all $z$ and $x$. It reads

$$\|P^n_0 - \pi\|_{TV} \geq \frac{1}{2} \lambda^n \approx \frac{1}{2} \exp(-\gamma n) = \frac{1}{2} \exp \left( -\frac{nm}{(m-1)d} \right),$$

which is small for $n \geq cd$ already, so the two bounds do not match.

e*) A tighter upper bound on the total variation distance can be found via the following analysis:

$$\|P^n_0 - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{z \in S \setminus \{0\}} \lambda^n_z} = \frac{1}{2} \sqrt{\sum_{t=1}^{d} \sum_{z \in S : |z| = t} \left( 1 - \frac{tm}{(m-1)d} \right)^{2n}}$$

As

$$\sum_{z \in S : |z| = t} = \binom{d}{t} (m-1)^t \leq \frac{(m-1)d^t}{t!} \quad \text{and} \quad \left( 1 - \frac{tm}{(m-1)d} \right)^{2n} \leq \exp \left( -\frac{2tmn}{(m-1)d} \right) \leq \exp \left( -\frac{2tn}{d} \right)$$

we finally obtain

$$\|P^n_0 - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{d} \frac{1}{t!} \exp \left( -t \left( \frac{2n}{d} - \log((m-1)d) \right) \right)}$$

Taking now $n = \frac{d}{2} (\log((m-1)d) + c)$, we obtain

$$\|P^n_0 - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{\infty} \frac{1}{t!} \exp(-tc)} = \frac{1}{2} \sqrt{\exp(e^{-c}) - 1}$$

which can be made arbitrarily small by taking $c$ large. So finally, the upper bound on the mixing time is $O(d \max(\log m, \log d))$. 

2
2. Following what has been done in class, we obtain first

\[
\|P_n^0 - \pi\|_2 = \left( \sum_{y \in S} \left( \frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y} \right)^2 \right)^{1/2} = \left( \sum_{z \in S : z \neq 0} \lambda_z^{2n} (\phi_0^{(z)})^2 \right)^{1/2}
\]

\[
\geq \left( \sum_{k=1}^{\lfloor d/2 \rfloor} \binom{d}{k} \left( 1 - \frac{2k}{d+1} \right)^{2n} \right)^{1/2} \geq \sqrt{d} \left( 1 - \frac{2}{d+1} \right)^n
\]

by retaining only the term \( k = 1 \) in the above sum. Using now the fact that \( e^{-x} \simeq 1 - x \) for \( x \) small, we obtain further

\[
\|P_n^0 - \pi\|_2 \geq \exp \left( \frac{1}{2} \log d - \frac{2n}{d+1} \right) = \exp(c/2)
\]

for \( n = \frac{d+1}{4} (\log d - c) \). The above expression can therefore be made arbitrarily large by taking \( c > 0 \) arbitrarily large.