1. a) Let us write by convention that \( y \sim x \) if there exists a unique \( j \in \{1, \ldots, d\} \) such that \( y_j \neq x_j \). Observing that the described process is a random walk on the graph described by the relation \( \sim \), we deduce that the transition matrix of the chain is given by
\[
p_{xy} = \begin{cases} 
\frac{1}{(m-1)d} & \text{if } y \sim x \\
0 & \text{otherwise}
\end{cases}
\]
The chain is clearly irreducible, aperiodic and positive-recurrent, therefore ergodic. Its stationary distribution \( \pi \) is uniform (i.e. \( \pi_x = m^{-d} \forall x \in S \)), and the detailed balance equation is satisfied.

b) Assume that \( |z| = k \) and denote by \( A \) the set of indices \( j \in \{1, \ldots, d\} \) such that \( z_j \neq 0 \) (so that \( |A| = k \)). Then
\[
\left( P\phi^{(z)} \right)_x = \sum_{y \in S} p_{xy} \phi_y = \frac{1}{(m-1)d} \sum_{y \sim x} \exp (2\pi iy \cdot z/m)
\]
\[
= \frac{1}{(m-1)d} \sum_{j=1}^{d} \sum_{t=0: t \neq x_j}^{m-1} \exp \left( 2\pi i \left( \sum_{l=1: l \neq j}^{d} x lz_l + tz_j \right)/m \right) 
\]
\[
= \frac{1}{(m-1)d} \sum_{j=1}^{d} \exp \left( 2\pi i \left( \sum_{l=1: l \neq j}^{d} x lz_l \right)/m \right) \times \sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi iuz_j/m)
\]
Observe now that if \( z_j = 0 \), then
\[
\sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi iuz_j/m) = \sum_{u=0: u \neq x_j}^{m-1} 1 = m - 1 = (m - 1) \exp(2\pi ix_jz_j/m)
\]
while if \( z_j \neq 0 \), then
\[
\sum_{u=0: u \neq x_j}^{m-1} \exp(2\pi iuz_j/m) = \sum_{u=0}^{m-1} \exp(2\pi iuz_j/m) - \exp(2\pi ix_jz_j/m) = 0 - \exp(2\pi ix_jz_j/m)
\]
This finally gives
\[
\left( P\phi^{(z)} \right)_x = \frac{1}{(m-1)d} \left( \sum_{j \in A} (-1) \exp (2\pi ix \cdot z/m) + \sum_{j \in A^c} (m-1) \exp (2\pi ix \cdot z/m) \right)
\]
\[
= \frac{1}{(m-1)d} \left( -d \exp (2\pi ix \cdot z/m) + (d - k)m \exp (2\pi ix \cdot z/m) \right) = \frac{(d-k)m - d}{(m-1)d} \phi^{(z)}_x
\]
\[
= \left( 1 - \frac{km}{(m-1)d} \right) \phi^{(z)}_x
\]
The eigenvalue \( \lambda_z \) corresponding to \( \phi^{(z)} \) is therefore given by
\[
\lambda_z = 1 - \frac{|z|m}{(m-1)d}
\]
c) The second largest eigenvalue is equal to $1 - \frac{m}{(m-1)d}$, while the least eigenvalue is equal to $1 - \frac{1}{m-1}$. When $d > 2$ (remember also that by assumption, $m > 2$), the spectral gap is therefore determined by the second largest eigenvalue and equal to $\gamma = \frac{m}{(m-1)d}$. This leads to the following upper bound on the total variation distance:

$$\|P_0^n - \pi\|_{TV} \leq \frac{1}{2 \sqrt{\pi_0}} \exp(-\gamma n) = \frac{m^{d/2}}{2} \exp\left(-\frac{nm}{(m-1)d}\right),$$

which becomes small only when $n \geq cd^2 \log m$ for some constant $c > 0$.

d) The lower bound obtained in class applies here, as $|\phi_x^{(z)}|^2 = 1$ for all $z$ and $x$. It reads

$$\|P_0^n - \pi\|_{TV} \geq \frac{1}{2} \lambda_n \approx \frac{1}{2} \exp(-\gamma n) = \frac{1}{2} \exp\left(-\frac{nm}{(m-1)d}\right)$$

which is small for $n \geq cd$ already, so the two bounds do not match.

e*) A tighter upper bound on the total variation distance can be found via the following analysis:

$$\|P_0^n - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{z \in S \setminus \{0\}} \lambda_z^{2n}} = \frac{1}{2} \sqrt{\sum_{t=1}^{d} \sum_{z \in S : |z|=t} \left(1 - \frac{tm}{(m-1)d}\right)^{2n}}$$

As

$$\sum_{z \in S : |z|=t} = \binom{d}{t} (m-1)^t \leq \frac{(m-1)d^t}{t!} \quad \text{and} \quad \left(1 - \frac{tm}{(m-1)d}\right)^{2n} \leq \exp\left(-\frac{2tmn}{(m-1)d}\right) \leq \exp\left(-\frac{2tn}{d}\right)$$

we finally obtain

$$\|P_0^n - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{d} \frac{1}{t!} \exp\left(-t \left(\frac{2n}{d} - \log((m-1)d)\right)\right)}$$

Taking now $n = \frac{d}{2} (\log((m-1)d) + c)$, we obtain

$$\|P_0^n - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\sum_{t=1}^{\infty} \frac{1}{t!} \exp(-tc)} = \frac{1}{2} \sqrt{\exp(e^{-c}) - 1}$$

which can be made arbitrarily small by taking $c$ large. So finally, the upper bound on the mixing time is $O(d \max(\log m, \log d))$. 


2. Following what has been done in class, we obtain first

\[ \| P_0^n - \pi \|_2 = \left( \sum_{y \in S} \left( \frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y} \right)^2 \right)^{1/2} = \left( \sum_{z \in S : z \neq 0} \lambda_z^{2n} (\phi_0^{(z)})^2 \right)^{1/2} \]

\[ \geq \left( \sum_{k=1}^{\lfloor d/2 \rfloor} \binom{d}{k} \left( 1 - \frac{2k}{d+1} \right)^{2n} \right)^{1/2} \geq \sqrt{d} \left( 1 - \frac{2}{d+1} \right)^n \]

by retaining only the term \( k = 1 \) in the above sum. Using now the fact that \( e^{-x} \approx 1 - x \) for \( x \) small, we obtain further

\[ \| P_0^n - \pi \|_2 \geq \exp \left( \frac{1}{2} \log d - \frac{2n}{d+1} \right) = \exp(c/2) \]

for \( n = \frac{d+1}{4} (\log d - c) \). The above expression can therefore be made arbitrarily large by taking \( c > 0 \) arbitrarily large.