1. “Only if” part: Let us fix $n$ and the sequence of $j_1, j_2, ..., j_n$. By using the detailed balance equation, we have

$$\pi_{j_1} p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_{n-1} j_n} \pi_{j_n} p_{j_n j_1}$$

By repeating the same set of calculations for the expression $p_{j_1 j_2} \cdots p_{j_{n-1} j_n} p_{j_n j_1}$ over the sequences of $j_2, \ldots, j_{n-1}$ as

$$\sum_{j_2, \ldots, j_{n-1}} p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n}$$

$$= p_{j_1 j_2} \sum_{j_2, \ldots, j_{n-1}} \mathbb{P}(X_n = j_n, X_{n-1} = j_{n-1}, \ldots, X_2 = j_2 | X_1 = j_1)$$

$$= p_{j_1 j_2} \mathbb{P}(X_n = j_n | X_1 = j_1) = p_{j_1 j_2} p_j^{(n-1)}$$

By repeating the same set of calculations for the expression $p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_{n-2} j_{n-1}} p_{j_{n-1} j_1}$, using the assumed equality, and considering the case for which $j_1 = i$ and $j_n = k$ (i.e., fixing the first and the last state in the sequence), we have

$$p_{ki} p_k^{(n-1)} = p_{ik} p_i^{(n-1)}$$

Since the chain is ergodic, as $n$ goes to $+\infty$, $p_i^{(n-1)}$ and $p_k^{(n-1)}$ go to $\pi_k$ and $\pi_i$ respectively. Therefore, we have

$$p_{ki} \pi_k = p_{ik} \pi_i$$

which is the detailed balance equation.
2. a) This chain is clearly ergodic. The transition matrix is

\[
\begin{pmatrix}
1 - p & p & 0 \\
1/2 & 0 & 1/2 \\
0 & p & 1 - p
\end{pmatrix}
\]

Assume that the detailed balance equation is satisfied. Then

\[
\pi^*_1/2 = \pi^*_2 p = \pi^*_0 p
\]

We conclude that

\[
\pi^*_0 = \pi^*_2 = \frac{1}{2(1 + p)} \quad \pi^*_1 = \frac{p}{1 + p}
\]

It is then easy to verify that \( \pi^* = \pi^* P \), and so this is indeed a stationary distribution, which obviously satisfies the detailed balance equation.

b) We know that \( \lambda_0 = 1 \), and so, to compute the eigenvalues, we must solve the equations

\[
2 - 2p = 1 + \lambda_1 + \lambda_2 \\
-p(1 - p) = \lambda_1 \lambda_2
\]

Solving this, we obtain that \( \lambda_1 = 1 - p \) and \( \lambda_2 = -p \). So \( \lambda_* = \max(p, 1 - p) \) and the spectral gap is given by \( \gamma = 1 - \lambda_* = \min(p, 1 - p) \).

c) For \( p = \frac{1}{N} \), the spectral gap is \( \gamma = \frac{1}{N} \). From the theorem seen in class, we know that \( \|P^n - \pi\|_{TV} \leq \exp(-\gamma n) \), so here,

\[
\max_{i \in S} \|P^n_i - \pi\|_{TV} \leq \frac{1}{2} \sqrt{\frac{1 + 1/N}{1/N}} \exp(-n/N) \leq \sqrt{N} \exp(-n/N) = \exp\left(\frac{\log N}{2} - \frac{n}{N}\right)
\]

Taking therefore \( n \geq N \left(\frac{\log N}{2} + c\right) \) with \( c > 0 \) sufficiently large (more precisely, \( c = \log(1/\varepsilon) \)) ensures that the maximum total variation norm is below \( \varepsilon \).

d) For \( p = 1 - \frac{1}{N} \), the spectral gap is again \( \gamma = \frac{1}{N} \). So

\[
\max_{i \in S} \|P^n_i - \pi\|_{TV} \leq \frac{1}{2} \sqrt{2(2 - 1/N)} \exp(-n/N) \leq \exp(-n/N)
\]

Taking therefore \( n \geq cN \) with \( c = \log(1/\varepsilon) \) ensures that the maximum total variation norm is below \( \varepsilon \), so

\[
T_\varepsilon \leq N \log(1/\varepsilon)
\]
3. a) The transition matrix being doubly stochastic, the stationary distribution is uniform (i.e. \( \pi_i = \frac{1}{2N} \) for every \( i \in S \)) and satisfies the detailed balance equation.

b) Solving the equation \( P \phi^{(1)} = \lambda \phi^{(1)} \), we obtain
\[
\begin{align*}
\frac{N-1}{N} a + \frac{1}{N} b &= \lambda a \\
\frac{N-1}{N} a - \frac{1}{N} b &= \lambda b
\end{align*}
\]
which is saying that \( \lambda \) is an eigenvalue of the \( 2 \times 2 \) matrix
\[
\begin{pmatrix}
\frac{N-1}{N} & \frac{1}{N} \\
-\frac{1}{N} & -\frac{1}{N}
\end{pmatrix}
= \begin{pmatrix} 1 - \delta & \delta \\ 1 - \delta & -\delta \end{pmatrix}
\]
where we have set \( \delta = \frac{1}{N} \). These eigenvalues are given by
\[
\lambda_{\pm} = 1 - 2\delta \pm \sqrt{(1 - 2\delta)^2 + 8\delta(1 - \delta)} = 1 - 2\delta \pm \sqrt{1 + 4\delta - 4\delta^2}
\]
For \( \delta \) small (i.e. \( N \) large), the largest of these 2 eigenvalues is \( \lambda_+ \), which is approximately given by
\[
\lambda_+ \simeq 1 - 2\delta + \frac{1}{2} \left( 1 + 2\delta - 4\delta^2 \right) = 1 - 2\delta^2 = 1 - \frac{2}{N^2}
\]
so the spectral gap \( \gamma \simeq \frac{2}{N^2} \).

c) By the theorem seen in class,
\[
\max_{i \in S} \| P_i^n - \pi \|_{TV} \leq \frac{\sqrt{2N}}{2} \exp(-\gamma n) \leq \sqrt{2} \exp \left( \frac{\log N}{2} - \frac{2n}{N^2} \right)
\]
is below \( \varepsilon \) for \( n \geq \frac{N^2}{2} \left( \frac{\log N}{2} + \log(\sqrt{2}/\varepsilon) \right) \), so
\[
T_\varepsilon \leq \frac{N^2}{2} \left( \frac{\log N}{2} + \log(\sqrt{2}/\varepsilon) \right)
\]