Markov Chains and Algorithmic Applications - IC - EPFL

Homework 5

Exercise 1. Let $(X_n,\, n \geq 0)$ be an ergodic Markov Chain. Show that $(X_n,\, n \geq 0)$ is reversible *if and only if*

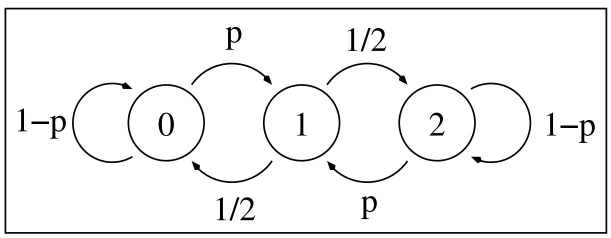
$$p_{j_1 j_2}\, p_{j_2 j_3} \cdots p_{j_{n-1} j_n}\, p_{j_n j_1} = p_{j_1 j_n}\, p_{j_n j_{n-1}} \cdots p_{j_3 j_2}\, p_{j_2 j_1}$$

for all $n \in \mathbb{N}$ and all finite sequences of states $j_1, j_2, ..., j_n$.

Hint 1. For the "only if" part: Fix n and the sequence of $j_1, j_2, \ldots, j_n$. Use the detailed balance equation to simplify the expression $\pi_{j_1}\, p_{j_1 j_2}\, p_{j_2 j_3} \cdots p_{j_{n-1} j_n}\, p_{j_n j_1}$.

Hint 2. For the "if" part: Fix n, and marginalize the mentioned equation over the sequences of $j_2, \ldots, j_{n-1}$. Then, consider the case when n goes to $+\infty$.

Exercise 2. Consider the Markov chain with the following transition graph, where $0 < p < 1$:



a) Compute the stationary distribution π of the chain. Is the detailed balance equation satisfied?

b) Compute the spectral gap γ as a function p.

Hint: In order to compute the eigenvalues λ_0, λ_1, λ_2 of the 3×3 transition matrix P, you have two options:

- either compute the roots of the characteristic polynomial: $\det(P - \lambda I) = 0$;

- or notice that $\mathrm{Tr}(P) = \lambda_0 + \lambda_1 + \lambda_2$, $\det(P) = \lambda_0\, \lambda_1\, \lambda_2$ and remember that $\lambda_0 = 1$ in the present case.

c) For $p = \frac{1}{N}$ with N large, deduce an asymptotic upper bound on the mixing time

$$T_\varepsilon = \inf\{n \geq 1 : \max_{i \in S} \|P_i^n - \pi\|_{\mathrm{TV}} \leq \varepsilon\}$$

for a given $\varepsilon > 0$.

d) Reproduce then the same computation for $p = 1 - \frac{1}{N}$ with N large.

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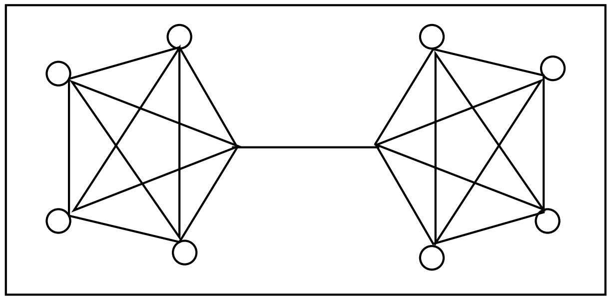
Exercise 3. Consider a graph made of two complete graphs, each of size N, which are linked by a single edge, as illustrated on the figure below for $N = 5$:



Note in addition that each vertex has a self-loop, except the two vertices which make the connection between the two graphs.

Consider now the Markov chain whose states are the vertices of the graph and whose transition probabilities are given by

$$p_{ij} = \begin{cases} 1/d_i & \text{if } i \text{ is connected to } j \\ 0 & \text{otherwise} \end{cases}$$

where d_i is the degree of vertex i. We are again interested in finding an asymptotic upper bound on the mixing time T_ε of the chain, using the spectral gap.

NB: Self-loops count for 1 in the degree of a vertex, so that all vertices here have degree N exactly.

a) Compute the stationary distribution π of the chain. Is the detailed balance equation satisfied?

b) In order to compute the spectral gap γ, you may use the following *hint*:

- As seen in class, it is always the case that the eigenvector $\phi^{(0)}$ associated with the largest eigenvalue $\lambda_0 = 1$ of the matrix P is the "all-ones" column vector $\phi_i^{(0)} = 1$ for every $i \in S$ (we do not normalize $\phi^{(0)}$ here).

- The eigenvector $\phi^{(1)}$ associated to the second largest eigenvalue $\lambda_1 < 1$ (which is orthogonal to $\phi^{(0)}$) can be shown here to be of the form

$$\phi^{(1)} = [\underbrace{+a, \ldots, +a}_{N-1 \text{ times}}, +b, -b, \underbrace{-a, \ldots, -a}_{N-1 \text{ times}}]^T, \quad \text{for some } a, b \in \mathbb{R}$$

and it can also be shown that the spectral gap γ is in this case equal to $\gamma = 1 - \lambda_1$ (not $1 - |\lambda_{N-1}|$).

c) Deduce from this an asymptotic upper bound on the mixing time of the chain:

$$T_\varepsilon = \inf\{n \geq 1 : \max_{i \in S} \|P_i^n - \pi\|_{\text{TV}} \leq \varepsilon\}$$

for a given $\varepsilon > 0$.