

Solutions 3

1. a) The chain is partitioned into two equivalence classes: 1. Non-negative numbers and odd negative numbers, and 2. Even negative numbers. Given that the state space is finite, we obtain after looking at the structure of the transition graph that the 1st class is positive recurrent, and the 2nd class is transient. So, using the first theorem in the lecture, the 1st class has a stationary distribution π_1 . Therefore, one possible option for π is to choose it in a way to be zero for the transient class and equal to π_1 for the 1st class, which is unique given the assumption of the exercise (proved at the end of the solution). So, it is sufficient to find π_1 over the first class, i.e., non-negative numbers and odd negative numbers.

Therefore, we have

$$\pi(i) = \begin{cases} \pi_1(i) & 0 \leq i \leq N, \\ \pi_1(-k) & i = -2k + 1 \text{ for a } k \in \{1, 2, \dots, K^+ = \lceil \frac{N}{2} \rceil\}, \\ 0 & i = -2k \text{ for a } k \in \{1, 2, \dots, K^- = \lfloor \frac{N}{2} \rfloor\}. \end{cases}$$

The stationary distribution must satisfy the following two 2nd order difference equations:

$$\begin{cases} \pi_1(-k) = \frac{1}{2}(\pi_1(-k-1) + \pi_1(-k+1)), & k \in \{1, 2, \dots, K^+ - 1\}, \\ \pi_1(k) = \frac{1}{2}(\pi_1(k-1) + \pi_1(k+1)), & k \in \{2, \dots, N-1\}, \end{cases}$$

which has a unique solution as

$$\begin{cases} \pi_1(-k) = A^- + B^-k, & k \in \{1, 2, \dots, K^+ - 1\}, \\ \pi_1(k) = A^+ + B^+k, & k \in \{2, \dots, N-1\}, \end{cases}$$

where A^- , A^+ , B^- , and B^+ should be found by satisfying the following boundary conditions.

1. Left end:

$$\begin{cases} \pi_1(-K^+) = \frac{1}{2}(\pi_1(-K^+) + \pi_1(-K^+ + 1)), \\ \pi_1(-K^+ + 1) = \frac{1}{2}(\pi_1(-K^+) + \pi_1(-K^+ + 2)). \end{cases}$$

2. Right end:

$$\begin{cases} \pi_1(N) = \frac{1}{2}(\pi_1(N) + \pi_1(N-1)), \\ \pi_1(N-1) = \frac{1}{2}(\pi_1(N) + \pi_1(N-2)). \end{cases}$$

3. Origin:

$$\begin{cases} \pi_1(-1) = \frac{1}{2}(\pi_1(-2) + \pi_1(0)), \\ \pi_1(0) = \frac{1}{2}\pi_1(1), \\ \pi_1(1) = \frac{1}{2}(\pi_1(-1) + \pi_1(0) + \pi_1(1)), \\ \pi_1(2) = \frac{1}{2}(\pi_1(1) + \pi_1(3)). \end{cases}$$

The conditions for the two ends leads to the equalities $\pi_1(N-1) = \pi_1(N-2)$ and $\pi_1(-K^+ + 1) = \pi_1(-K^+ + 2)$, which leads to $B^+ = B^- = 0$. The conditions for the intersection in the origin, leads a relation between A^- and A^+ as $A^+ = 2A^-$. As a result, the stationary distribution is

$$\begin{cases} \pi_1(-k) = q, & k \in \{0, 1, \dots, K^+\}, \\ \pi_1(k) = 2q, & k \in \{1, 2, \dots, N\}. \end{cases}$$

and as a result

$$\pi(i) = \begin{cases} 2q & 1 \leq i \leq N, \\ q & i = 0, \\ q & i = -2k + 1 \text{ for a } k \in \{1, 2, \dots, K^+ = \lceil \frac{N}{2} \rceil\}, \\ 0 & i = -2k \text{ for a } k \in \{1, 2, \dots, K^- = \lfloor \frac{N}{2} \rfloor\}. \end{cases}$$

where $q = A^-$ should be found in a way to satisfy the condition that the sum of probabilities is 1, as

$$q = \frac{1}{2N + K^+ + 1}.$$

b) The probability of being on the negative part is as

$$\sum_{-N \leq k < 0} \pi(k) = K^+ q = \frac{K^+}{2N + K^+ + 1}$$

which is roughly equal to 0.2 for $N \gg 1$.

Bonus part of a) Let us consider indexes $\{1, 2, \dots, N + K^+ + 1\}$ for the states in the first equivalence class C_1 , and indexes $\{N + K^+ + 2, \dots, 2N + 1\}$ for the states in the equivalence class C_2 . Then, the transition probability matrix for this chain can be written as

$$P = \left[\begin{array}{c|c} P_{C_1 C_1} & P_{C_1 C_2} \\ \hline P_{C_2 C_1} & P_{C_2 C_2} \end{array} \right].$$

Since there is no path from C_1 to C_2 , the matrix $P_{C_1 C_2}$ is a zero matrix. Let us also show the stationary distribution of this chain with $\pi = [\pi_1, \pi_2]$. To be a stationary probability distribution, π should be a left eigenvector of P with eigenvalue equal to 1, which leads to

$$\begin{aligned} \pi_1 &= \pi_1 P_{C_1 C_1} + \pi_2 P_{C_2 C_1}, \\ \pi_2 &= \pi_2 P_{C_2 C_2}. \end{aligned}$$

Since C_2 is a transient equivalence class and $P_{C_2 C_2}$ is not a proper transition probability matrix (the summation of elements of some of its rows is not equal to 1), $P_{C_2 C_2}$ does not have any eigenvalue equal to 1. Therefore, the mentioned equations can be simplified as

$$\begin{aligned} \pi_1 &= \pi_1 P_{C_1 C_1}, \\ \pi_2 &= 0_{1 \times K^-}. \end{aligned}$$

Since C_1 has a unique stationary distribution (according to theorem 1), π is unique as

$$\pi = [\pi_1, 0_{1 \times K^-}],$$

where π_1 is the unique probability distribution of C_1 . This is exactly what we found for the chain in part **a)**.

2. a) First observe that the transition probabilities do not depend on the particular shape of the (convex) polygon, but just on the number of edges. Consider next a polygon with $j + 3$ edges initially. After the transition, the smallest possible polygon will have 3 edges and the largest possible polygon will have $j + 4$ edges. Thus the resulting polygons have $k + 3$ edges with $0 \leq k \leq j + 1$. Since the transition is uniformly random, $p_{jk} = \frac{1}{j+2}$, for $0 \leq k \leq j + 1$.

b) Thus,

$$\mathbb{E}(X_n | X_{n-1} = j) = \sum_{k=0}^{j+1} k p_{jk} = \frac{1}{j+2} \sum_{k=0}^{j+1} k = \frac{(j+1)(j+2)}{2(j+2)} = \frac{j+1}{2}$$

and

$$\mathbb{E}(X_n) = \sum_{j \geq 0} \mathbb{E}(X_n | X_{n-1} = j) \mathbb{P}(X_{n-1} = j) = \frac{1 + \mathbb{E}(X_{n-1})}{2}$$

Repeating this, we obtain $\mathbb{E}(X_n) = 1 - (1/2)^n + (1/2)^n \mathbb{E}(X_0)$.

c) Consider $(X_n, n \geq 0)$ initialized with some initial distribution for X_0 . Repeating the above computation, we obtain

$$\mathbb{E}(s^{X_n} | X_{n-1} = j) = \frac{1}{j+2} \sum_{k=0}^{j+1} s^k = \frac{1}{j+2} \frac{1 - s^{j+2}}{1 - s}$$

This implies that $G_n(s) = \frac{1}{1-s} \mathbb{E} \left(\frac{1 - s^{(X_{n-1}+2)}}{X_{n-1} + 2} \right)$.

d) Now consider the process $(X_n, n \geq 0)$ initialized with $X_0 \sim \pi$, where π is the stationary distribution. Since $\pi = \pi P$ by definition, we have $X_n \sim \pi$ and $X_{n-1} \sim \pi$, so $G(s) = \mathbb{E}(s^{X_n}) = \mathbb{E}(s^{X_{n-1}})$ for all $n \geq 1$, and by part 3, we also have

$$G(s) = \frac{1}{1-s} \mathbb{E} \left(\frac{1 - s^{(X_{n-1}+2)}}{X_{n-1} + 2} \right)$$

where $\mathbb{E}$ is taken with respect to π .

Differentiating with respect to s , we obtain

$$\begin{aligned} G'(s) &= \frac{1}{(1-s)^2} \mathbb{E} \left(\frac{1 - s^{X_{n-1}+2}}{X_{n-1} + 2} \right) - \frac{1}{1-s} \mathbb{E} \left(\frac{X_{n-1} + 2}{X_{n-1} + 2} s^{X_{n-1}+1} \right) \\ &= \frac{1}{1-s} G(s) - \frac{1}{1-s} s G(s) = G(s) \end{aligned}$$

One checks also that $G(1) = 1$ (using Bernoulli-L'Hospital's rule), so the solution of this differential equation is $G(s) = e^{s-1}$.

e) $G_Y(s) = \sum_{k \geq 0} \lambda^k e^{-\lambda} s^k / k! = e^{-\lambda} \sum_{k \geq 0} (s\lambda)^k / k! = e^{\lambda(s-1)}$. Hence the stationary distribution of the Markov chain is Poisson with parameter 1.

3. a) Clearly, all states i s.t. i is not a power of 2 are transient. Indeed, if i is not a power of 2, then $\mathbb{P}(X_n = i \text{ for some } n > 0 | X_0 = i) = 0$.

Let us consider now the state 1. Then,

$$f_{11}(n) = c^{n-1}(1-c), \quad \text{for } n \geq 1.$$

Hence,

$$f_{11} = \sum_{n \geq 1} f_{11}(n) = (1 - c) \sum_{n \geq 1} c^{n-1} = \frac{1 - c}{1 - c} = 1$$

and

$$\mu_1 = \sum_{n \geq 1} n f_{11}(n) = (1 - c) \sum_{n \geq 1} n c^{n-1} = \frac{1}{1 - c} < +\infty,$$

which implies that the state 1 is positive-recurrent. As concerns the states $\{2^k\}_{k \geq 1}$, they form together with state 1 an equivalence class of the Markov chain. Therefore, they are also positive-recurrent.

b) Let π be the stationary distribution (in case it exists). Then, by solving $\pi = \pi P$, we obtain

$$\begin{cases} \pi_1 = (1 - c) \sum_{i \in \mathbb{N}} \pi_{2^i} = 1 - c \\ \pi_{2^k} = c \cdot \pi_{2^{k-1}} & k \geq 1 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

Hence, the stationary distribution exists, is unique and is given by

$$\begin{cases} \pi_{2^k} = (1 - c) \cdot c^k & k \geq 0 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

c) In general, by solving $\pi = \pi P$, we obtain

$$\begin{cases} \pi_1 = \sum_{k \in \mathbb{N}} (1 - p_{2^k}) \pi_{2^k} \\ \pi_{2^k} = p_{2^{k-1}} \cdot \pi_{2^{k-1}} & k \geq 1 \\ \pi_i = 0 & \text{otherwise} \end{cases}$$

Therefore, $\pi_{2^k} = \prod_{j=0}^{k-1} p_{2^j} \pi_1$, so the stationary distribution exists and is unique if and only if

$$\sum_{k \in \mathbb{N}} \prod_{j=0}^{k-1} p_{2^j} < +\infty. \quad (1)$$

(otherwise it would imply that $\pi_1 = 0$).

d) Consider now the case $c_k = p_{2^k} = 1 - \frac{1}{2^k + 1}$. We will show that $\lim_{k \rightarrow \infty} \prod_{j=0}^k c_j \neq 0$, which implies, through condition (1), that the stationary distribution does not exist. Note first that

$$\lim_{k \rightarrow \infty} \prod_{j=0}^k c_j = 0 \quad \Longleftrightarrow \quad \lim_{k \rightarrow \infty} \sum_{j=0}^k \log \frac{1}{c_j} = +\infty.$$

In addition,

$$\lim_{k \rightarrow \infty} \sum_{j=0}^k \log \frac{1}{c_j} = \lim_{k \rightarrow \infty} \sum_{j=0}^k \log (1 + 2^{-j}) \leq \lim_{k \rightarrow \infty} \sum_{j=0}^k 2^{-j} = 2 < +\infty,$$

where we used the fact that $\log(1 + x) \leq x$ for any $x \in [0, 1]$. As a result, $\sum_{k \in \mathbb{N}} \prod_{j=0}^{k-1} c_j = +\infty$, so the stationary distribution does not exist in this case.