Markov Chains and Algorithmic Applications - IC - EPFL

Homework 3

Exercise 1. Consider the time-homogeneous Markov chain $(S_n, n \geq 0)$ with state space $S = \{-N, ..., N\}$ (for some fixed value N) and transition probabilities

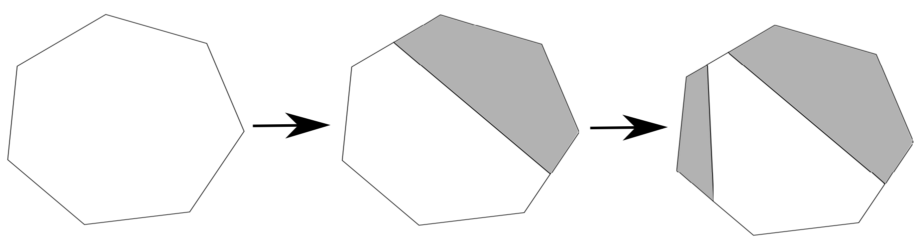
$$p_{-N,-N} = p_{-N,-N+2} = 0.5, \quad p_{-N+1,-N+1} = p_{-N+1,-N+3} = 0.5, \quad p_{N,N} = p_{N,N-1} = 0.5$$

$$p_{ij} = \begin{cases} 0.5 \quad \text{if} \quad -N+2 \leq i < 0 \text{ and } (j = i-2 \text{ or } j = i+2) \\ 0.5 \quad \text{if} \quad 0 \leq i \leq N-1 \text{ and } (j = i-1 \text{ or } j = i+1) \\ 0 \quad \text{otherwise} \end{cases}$$

a) Compute the unique stationary distribution π of the chain.
Bonus: Show that it exists and is unique.

b) Compute $\sum_{j=-N}^{0} \pi_j$ (corresponding to the average proportion of time spent by the chain on the negative axis).

Exercise 2. A random sequence of convex polygons is generated by picking two edges of the current polygon at random, joining their midpoints, and picking one of the two resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number number of edges of the n^{th} polygon thus constructed (so that X_n takes values in the nonnegative integers; for example, $X_n = 0$ corresponds to a triangle, $X_n = 1$ to a quadrilateral, etc).



The white part is selected at each step.

a) What are the transition probabilities associated to the Markov chain $(X_n,\ n \geq 0)$?

b) Compute $\mathbb{E}(X_n)$ in terms of X_0. *Hint*: Compute first $\mathbb{E}(X_n | X_{n-1} = j)$.

c) We define the probability generating function of the process $(X_n, n \geq 0)$ by $G_n(s) = \mathbb{E}(s^{X_n})$. Prove that

$$G_n(s) = \frac{1}{1-s}\,\mathbb{E}\left(\frac{1 - s^{(X_{n-1}+2)}}{X_{n-1}+2}\right) \quad \text{for } n \geq 1$$

d) Now suppose that the process $(X_n, n \geq 0)$ is initialized with $X_0 \sim \pi$, where π is the stationary distribution. Argue that $\mathbb{E}(s^{X_n})$ is independent of n with this initialization. Prove that $G'(s) = G(s)$, $G(1) = 1$, and solve this differential equation.

e) Compute the probability generating function of a Poisson distribution Y with parameter λ, $\mathbb{P}(Y = k) = \lambda^k e^{-\lambda}/k!$. Conclude that the stationary distribution is Poisson.

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Exercise 3. Let $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain on $S = \{1, 2, 3, \ldots\}$ with transition probabilities:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} p_i & \text{if } j = 2i, \\ 1 - p_i, & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < p_i < 1$ are arbitrary numbers in general (note in particular that we do *not* ask that $\sum_{i \geq 1} p_i = 1$).

Let us start with the simple case where $p_i \equiv c$ for some $0 < c < 1$.

a) Which states are transient, which are mull-recurrent, which are positive-recurrent? Justify your answer!

b) Does the chain admit a stationary distribution? If yes, compute it!

Consider now the general case where the probabilities p_i are arbitrary numbers between 0 and 1.

c) Find an optimal condition on the numbers p_i ensuring the existence (and uniqueness) of a stationary distribution.

d) In the particular case where $p_i = 1 - \frac{1}{i+1}$ for $i \geq 1$, does the chain admit a (unique) stationary distribution? If yes, compute it!