Stationary distribution

Distribution at time $n$: $\pi_j^{(n)} = P(X_n = j) \, \forall n \in \mathbb{N}, j \in S$

\[
\pi_j^{(n+1)} = \sum_{i \in S} \pi_i^{(n)} P_{ij} \quad \forall j \in S
\]

In vector form: \(\pi^{(n+1)} = \pi^{(n)} \cdot P\)

raw vector \quad raw vector matrix

**Definition**: A (probability) distribution \(\pi = (\pi_i, i \in S)\)

\([0 \leq \pi_i \leq 1, \sum_{i \in S} \pi_i = 1]\) is a **stationary distribution** for the

Markov chain \(X\) if \(\pi_j = \sum_{i \in S} \pi_i P_{ij} \quad \forall j \in S\)

i.e. \(\pi = \pi \cdot P\)
Implications:
- If $\pi$ is stationary, then $\pi \cdot P^n = \pi \cdot P \cdot P^{n-1} = \pi P^{n-1} = \cdots = \pi$
- If $\pi^{(0)} = \pi$ (stat. dist.), then $\pi^{(n)} = \pi^{(0)} \cdot P^n = \pi \cdot P^n = \pi$ $\forall n \in \mathbb{N}$

Remarks:
- A stationary distribution is a solution of a system of linear equations; it is not necessarily the case that $\lim_{n \to \infty} \pi^{(n)} = \pi$
- $\pi$ may not exist in some cases
- $\pi$ may not be unique in some other cases
- Practical remark: in the system of $N$ equations $\pi = \pi \cdot P$ (assume $|S| = N$), there is always one redundant equation; in order to determine $\pi$, we need to use also the condition $\sum_{i \in S} \pi_i = 1$. 
Mathematical remark:

Define \( \mathbf{1} \) = “all-ones” column vector

Then \( P \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{1} \) (i.e. \( \sum_{j \in S} p_{ij} = 1 \) \( \forall i \in S \))

“Stochastic matrix”

So \( \mathbf{1} \) is an eigenvector of the matrix \( P \) (on the right) with corresponding eigenvalue \( 1 \).

Now if there exists a row vector \( \mathbf{T} \) s.t. \( \mathbf{T} = \mathbf{T} \cdot P \), then \( \mathbf{T} \) is also an eigenvector of \( P \) (on the left) with the same eigenvalue \( 1 \).
Theorem [without proof]

Let $X$ be an irreducible Markov chain. Then $X$ is positive-recurrent $\iff$ $X$ admits a stationary distribution $\pi$. In addition, in this case, if $\pi$ exists, then it is unique and given by $\pi_i = \frac{1}{\mu_i} = \frac{1}{\sum (T_i(X_0 = i))}$ for all $i \in S$.

Note: $X$ is positive-recurrent $\Rightarrow \mu_i < \infty$, so $\pi_i > 0$ for all $i \in S$.

Corollary: A finite irreducible chain always admits a unique stationary distribution.
Example

Cyclic random walk

\[ S = \{0, \ldots, N-1\} \]

finite, irreducible

\[ \Rightarrow \text{positive-recurrent} \]

Then

\[ \exists \text{ exists } & \text{is unique} \]

\[ P = \begin{pmatrix}
0 & \ldots & 0 & q \\\nq & \ldots & q & 0 \\
0 & \ldots & 0 & q \\
p & \ldots & q & 0 \\
\end{pmatrix} \]

\[ p+q=1 \]
\[ 0 < p, q < 1 \]

\[ \sum_{i \in S} p_{ij} = 1 \quad \forall i \in S \]

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"doubly stochastic matrix"
Proposition

If $X$ is a finite irreducible chain whose transition matrix $P$ is doubly stochastic, then it admits a unique stationary distribution $\pi$ and $\pi$ is uniform: $\pi_i = \frac{1}{N}$ $\forall i \in S$ ($|S|=N$).

Proof: Plug $\pi_i = \frac{1}{N}$ into the equation $\pi = \pi P$.

$$\frac{1}{N} = \sum_{i \in S} \frac{1}{N} P_{ij} \quad \forall i \in S?$$

$$1 = \sum_{i \in S} P_{ij} \quad \forall i \in S? \quad \checkmark \quad \text{because } P \text{ is doubly stochastic}$$

(uniqueness guaranteed by the theorem)
Back to the example

- So \( \pi_i = \frac{1}{N} \) \( \forall i \in \{0, \ldots, N-1\} \)

The thin also says that \( \pi_i = \frac{1}{\mu_i} \) so \( \mu_i = N \forall i \)

- So \( \pi \) uniform is the "stationary" distribution of the chain

\[ \uparrow \]

when \( p \neq q \), a rotation occurs permanently in one direction or the other => not "truly" stationary
Counter-example

Symmetric simple random walk on \( \mathbb{Z} \):

irreducible, recurrent but null-recurrent

Let us prove that the chain is null-recurrent using the theorem: look for a stationary distribution \( \pi \):

\[ \pi = \pi P \quad \text{i.e.} \quad \forall i \in \mathbb{Z} \quad \pi_i = \frac{1}{2}(\pi_{i+1} + \pi_{i-1}) \]

\[ \Rightarrow \quad \pi_i = \pi_j \quad \forall i, j \in \mathbb{Z} \quad \rightarrow \text{problem!} \]

The uniform distribution does not exist on \( \mathbb{Z} \)!

\[ \Rightarrow \quad \pi \text{ does not exist} \quad \Rightarrow \quad X \text{ is not positive-recurrent} \]

Thus

\[ \Rightarrow \quad X \text{ is null-recurrent. \#} \]
What if the chain is not irreducible?

- Two positive-recurrent classes:

  \[ \Rightarrow \text{a stationary distribution exists} \]
  \[ \text{but is not unique!} \]

  \[ \begin{align*}
  \pi^{(1)} &= \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right), \\
  \pi^{(2)} &= \left( 0, 0, \frac{1}{2}, \frac{1}{2} \right), \\
  \pi^{(3)} &= \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)
  \end{align*} \]

  \[ \begin{align*}
  \pi &= \left( \frac{\alpha}{2}, \frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2} \right), \\
  0 \leq \alpha \leq 1 \text{ are stationary distributions of the chain}
  \end{align*} \]
• Two positive-recurrent classes and one transient class:

\[ \Pi \text{ exists but is not unique: } \Pi = \left( \frac{x}{2}, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{1-x}{2} \right) \]

• Two transient classes and one positive-recurrent class:

\[ \Pi \text{ exists and is unique: } \Pi = (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0) \]
Limiting distribution

Definition: A distribution \( \pi \) is a limiting distribution for the Markov chain \( (x_n, n \geq 0) \) if

for any initial distribution \( \pi^{(0)} \), \( \lim_{n \to \infty} \pi^{(n)} = \pi \)

Remarks:

- such a limiting distribution may not exist
- but if it exists, then it is unique!
- if \( \pi \) is a limiting distribution, then it is a stationary dist.

Proof: \( \pi^{(n+1)} = \pi^{(n)}. P \) \( \forall n \in \mathbb{N} \)

\( \lim_{n \to \infty} \pi^{(n)} = \pi. P \)

"#" (\( \Delta |S| = \pm \infty \) case)
Def: A Markov chain is **ergodic** if it is irreducible, aperiodic and positive-recurrent.

**Ergodic theorem**

Let $X$ be an ergodic Markov chain. Then it admits a unique limiting and stationary distribution $\pi$, i.e.:

- $\forall \pi^{(0)}$, $\lim_{n\to\infty} \pi^{(n)} = \pi$
- $\pi = \pi P$

$\forall i,j \in S$

$\lim_{n\to\infty} P(X_n = j | X_0 = i) = \pi_j$
Remark: aperiodicity matters!

Ex: consider the chain

\[ p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

periodicity = 2

stationary distribution?

\[ \pi P \rightarrow \pi = \left( \frac{1}{2}, \frac{1}{2} \right) \]

is the solution

hunting distribution?

if \( \pi^{(0)} = (1, 0) \), then \( \pi^{(1)} = (0, 1) \), \( \pi^{(2)} = (1, 0) \) ... 

so \( \lim_{n \to \infty} \pi^{(n)} \) does not exist!
Modified ex: \( P = \begin{pmatrix} \frac{3}{1} & \frac{3}{1} \\ \frac{1}{1} & \frac{1}{1} \end{pmatrix} \) 

finite, reducible, aperiodic chain \( \Rightarrow \) ergodic

\[ \Rightarrow \exists! \ \pi = \text{limiting & stationary distribution} \]

Last remark: So can't we say anything for a periodic chain? Yes we can!

( irreducible & positive-recurrent)

\[ \forall \pi(0), \quad \frac{1}{n} \sum_{k=1}^{n} \pi_i(k) \xrightarrow{n \to \infty} \pi_i, \quad \forall i \in S \]