1 Recurrence and transience

Definition 1.1. •

- A state $i \in S$ is **recurrent** if $f_{ii} = \mathbb{P}(\exists n \geq 1 \text{ such that } X_n = i \mid X_0 = i) = 1$
  (i.e., the probability that the chain returns to state $i$ in finite time is equal to 1).

- A state $i \in S$ is **transient** if $f_{ii} < 1$.

So a state is recurrent if and only if it is not transient.
Note in particular that it is not necessary that $f_{ii} = 0$ for state $i$ to be transient.

Examples.

![Figure 1: Here states A, B and C are transient and D is recurrent.](image)

![Figure 2: For $0 < p, q < 1$ and $p + q = 1$, are the states transient or recurrent?](image)

Facts. •

- In a given equivalence class, either all states are recurrent, or all states are transient.

- In a finite chain, an equivalence class is recurrent iff there is no arrow leading out of it.
  (So a finite irreducible chain is always recurrent.)

- In a infinite chain, things are more complicated. (The chain might "escape to infinity").

In order to deal with infinite chains, we need to establish a relation between the following two sequences
of numbers:

- $p_{ii}^{(n)} = \mathbb{P}(X_n = i \mid X_0 = i)$ (with the convention $p_{ii}^{(0)} = 1$)
- $f_{ii}^{(n)} = \mathbb{P}(X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i)$ (with the convention $f_{ii}^{(0)} = 0$)

In words, $f_{ii}^{(n)}$ is the probability that, having left state $i$ at time 0, the chain returns to state $i$ at time $n$
for the first time.
Lemma 1.2. ∀n ≥ 1, we have:

\[ p_{ii}^{(n)} = \sum_{m=1}^{n} f_{ii}^{(m)} p_{ii}^{(n-m)} \]

Proof. Let

\[ A_n = \{ X_n = i \} : p_{ii}^{(n)} = \mathbb{P}(A_n | X_0 = i) \]
\[ B_n = \{ X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i \} : f_{ii}^{(n)} = \mathbb{P}(B_n | X_0 = i) \]

If the event \( A_n \) takes place, then it must be that one of the event \( B_1, \ldots, B_n \) also happen (because in the worst case, \( X \) will return to state \( i \) at time \( n \)). Therefore:

\[ p_{ii}^{(n)} = \mathbb{P}(A_n | X_0 = i) = \mathbb{P}(A_n \cap (\bigcup_{m=1}^{n} B_m) | X_0 = i) \]
\[ = \sum_{m=1}^{n} \mathbb{P}(A_n \cap B_m | X_0 = i) = \sum_{m=1}^{n} \mathbb{P}(A_n | B_m, X_0 = i) \mathbb{P}(B_m | X_0 = i) \]
\[ = \sum_{m=1}^{n} \mathbb{P}(X_n = i | X_m = i, X_{m-1} \neq i, \ldots, X_1 \neq i, X_0 = i) \mathbb{P}(X_m = i, X_{m-1} \neq i, \ldots, X_1 \neq i | X_0 = i) \]

where we have used the Markov property in the last equality leading to the term \( p_{ii}^{(n-m)} \).

Proposition 1.3. A state \( i \in S \) is recurrent iff \( \sum_{n \geq 1} p_{ii}^{(n)} = +\infty \).
(So a state \( i \in S \) is transient iff \( \sum_{n \geq 1} p_{ii}^{(n)} < +\infty \))

Proof. First note

\[ f_{ii} = \mathbb{P}(\exists n \geq 1 \mbox{ s.t. } X_n = i | X_0 = i) = \mathbb{P}(\bigcup_{n \geq 1} B_n | X_0 = i) = \sum_{n \geq 1} \mathbb{P}(B_n | X_0 = i) = \sum_{n \geq 1} f_{ii}^{(n)} \]

So what we need to prove is that \( \sum_{n \geq 1} f_{ii}^{(n)} = 1 \) iff \( \sum_{n \geq 1} p_{ii}^{(n)} = +\infty \).

Observe that there is a convolution relation between \( p_{ii}^{(n)} \)'s and \( f_{ii}^{(n)} \)'s. We will therefore use generating functions to get a simpler relation. Define for \( s \in [0, 1] \):

\[ P_{ii}(s) = \sum_{n \geq 0} s^n p_{ii}^{(n)} \quad \text{and} \quad F_{ii}(s) = \sum_{n \geq 0} s^n f_{ii}^{(n)} \]

We will need now the following fact, also known as Abel’s theorem:

Fact (Abel’s Theorem). Let \( (a_n, n \geq 0) \) be a sequence of numbers s.t. \( 0 \leq a_n \leq 1, \forall n \geq 0 \). Then, \( A(s) = \sum_{n \geq 0} s^n a_n \) converges \( \forall s, |s| < 1 \) and

\[ \lim_{s \to 1} A(s) = \sum_{n \geq 0} a_n \in \mathbb{R}_+ \quad \text{or} \quad \lim_{s \to 1} A(s) = \sum_{n \geq 0} a_n = +\infty \]

So for \( |s| < 1 \), we have:

\[ P_{ii}(s) = 1 + \sum_{n \geq 1} s^n p_{ii}^{(n)} = 1 + \sum_{n \geq 1} s^n \left( \sum_{m=1}^{n} f_{ii}^{(m)} p_{ii}^{(n-m)} \right) \]
\[ = 1 + \sum_{n \geq 1} \sum_{m=1}^{n} s^m s^{n-m} f_{ii}^{(m)} p_{ii}^{(n-m)} = 1 + \sum_{m \geq 1} \sum_{n \geq m} s^m f_{ii}^{(m)} s^{n-m} p_{ii}^{(n-m)} \]
\[ = 1 + \sum_{m \geq 1} s^m f_{ii}^{(m)} \sum_{k \geq 0} s^k p_{ii}^{(k)} = 1 + F_{ii}(s) P_{ii}(s) \]

remembering that \( f_{ii}^{(0)} = 0 \), by convention.
Hence, \( P_{ii}(s) = \frac{1}{1 - F_{ii}(s)} \) for all \(|s| < 1\) and by Abel’s theorem:
\[
\sum_{n \geq 0} p_{ii}^{(n)} = \lim_{s \to 1} P_{ii}(s) = +\infty \quad \text{iff} \quad f_{ii} = \sum_{n \geq 0} f_{ii}^{(n)} = \lim_{s \to 1} F_{ii}(s) = 1
\]

Remark.
\[
\sum_{n \geq 1} p_{ii}^{(n)} = \sum_{n \geq 1} \mathbb{P}(X_n = i | X_0 = i) = \text{expected number of visits of state } i \text{ | } X_0 = i
\]

So this expected number of visits of state \( i \) is infinite iff \( i \) is recurrent.

**Example 1.4.** - One-dimensional simple (a-)symmetric random walk: by Homework 1, Exercise 1:
\[
p_{00}^{(2n)} \approx \frac{(4pq)^n}{\sqrt{\pi n}} \quad \text{for } n \text{ large}
\]
The chain is recurrent iff state 0 is recurrent iff
\[
\sum_{n \geq 1} p_{00}^{(n)} = +\infty \quad \text{iff} \quad \sum_{n \geq 1} p_{00}^{(2n)} = +\infty \quad \text{iff} \quad \sum_{n \geq 1} \frac{(4pq)^n}{\sqrt{\pi n}} = \infty
\]
iff \( p = q = 1/2 \) (else \( 4pq < 1 \) and the series converges).

- Two-dimensional simple symmetric random walk (see Homework 1, Exercise 2):
\[
p_{00}^{(2n)} \approx \frac{1}{\pi n} \quad \text{for } n \text{ large}
\]
so \( \sum_{n \geq 1} p_{00}^{(2n)} = +\infty \) and the chain is recurrent.

- Three-dimensional simple symmetric random walk: see Homework 2, Exercise 2.

## 2 Positive and null-recurrence

Let \( T_i = \inf\{n \geq 1 : X_n = i\} \) be the first recurrence time to state \( i \). So \( f_{ii}^{(n)} = \mathbb{P}(T_i = n | X_0 = i) \) and
\[
f_{ii} = \sum_{n \geq 1} f_{ii}^{(n)} = \sum_{n \geq 1} \mathbb{P}(T_i = n | X_0 = i) = \mathbb{P}(T_i < +\infty | X_0 = i) \begin{cases} = 1 \text{ iff } i \text{ is recurrent} \\ < 1 \text{ iff } i \text{ is transient} \end{cases}
\]

**Definition 2.1.** The **mean recurrence time** to state \( i \) is defined as \( \mu_i = \mathbb{E}(T_i | X_0 = i) \) •

- if \( i \) is transient, then \( \mathbb{P}(T_i = +\infty | X_0 = i) > 0 \), so \( \mu_i = +\infty \).

- if \( i \) is recurrent, then \( \mu_i = \sum_{n \geq 1} n \mathbb{P}(T_i = n | X_0 = i) \geq 0 \in [1, +\infty] \).

In this case, we say that •

- \( i \) is **positive-recurrent** if \( \mu_i < +\infty \).

- \( i \) is **null-recurrent** if \( \mu_i = +\infty \).
Remarks. ●

- What does is mean to be recurrent? By time-homogeneity, this implies that the chain will visit state \( i \) an infinite number of times with probability 1.

- In the case of a positive-recurrent state, the average time duration between two visits is finite.

- In the case of a null-recurrent state, this average time duration between two visits is infinite, but the probability to return in finite time is 1, as counter-intuitive as it may be!

Facts. ●

- In a given equivalence class, either all states are transient, or all state are positive-recurrent, or all states are null-recurrent.

- A finite irreducible chain is always positive-recurrent.

Example.

\[ \cdots \xrightarrow{p} -2 \xrightarrow{p} -1 \xrightarrow{p} 0 \xrightarrow{p} 1 \xrightarrow{p} 2 \xrightarrow{p} \cdots \]

- \( p \neq q \implies \) transient chain \( \implies \mathbb{P}(T_0 = +\infty|X_0 = 0) > 0 \) and \( \mu_0 = +\infty \)

- \( p = q = \frac{1}{2} \implies \) recurrent chain \( \implies \mathbb{P}(T_0 = +\infty|X_0 = 0) = 0 \), but \( \mu_0 = +\infty \) also (without proof); the chain is null-recurrent in this second case.