Martingales

Def: Let $(\Omega, F, P)$ be a probability space.

A filtration is a sequence of sub-$\sigma$-fields of $F$ $(F_n, n \in \mathbb{N})$

such that $F_n \subseteq F_{n+1}$, $\forall n \in \mathbb{N}$.

Ex: $\Omega = [0, 1]$ , $F = B([0, 1])$ , $P =$ lebesgue measure on $[0,1]$

$X_n(\omega) =$ $n^{th}$ decimal of $\omega \in [0, 1]$ $\forall n \geq 1$

$F_0 = \{ \emptyset, \Omega \} , F_1 = \sigma(X_1) , F_2 = \sigma(X_1, X_2) \ldots F_n = \sigma(X_1, \ldots, X_n)$ $\forall n \geq 1$

$\omega = 0.17392 \ldots$
Def: Let \((F_n, n \in \mathbb{N})\) be a filtration on \((\Omega, \mathcal{F}, P)\). 

A square-int. martingale w.r.t \((F_n, n \in \mathbb{N})\) is a discrete-time stochastic process \((M_n, n \in \mathbb{N})\) such that:

(i) \(E(M_n^2) < +\infty\) \(\forall n \in \mathbb{N}\) 

(ii) \(M_n\) is \(F_n\)-measurable \(\forall n \in \mathbb{N}\) (adapted process) 

\[ \rightarrow \quad \text{(iii) } E(M_{n+1} | F_n) = M_n \text{ a.s. } \forall n \in \mathbb{N} \]

Rmk: Because \(E(M_{n+1} | F_n)\) is \(F_n\)-measurable by definition, condition (ii) is actually redundant.

Rmk: martingale \(\neq\) Markov process
Example: The simple symmetric random walk.

Let \((X_n, n \geq 1)\) be iid r.v.'s st. \(P(\{X_i = +1\}) = P(\{X_i = -1\}) = \frac{1}{2}\)

\(S_0 = 0, \ S_n = X_1 + \ldots + X_n \quad n \geq 1\)

\(\mathcal{F}_0 = \{\emptyset, \Omega\}, \ \mathcal{F}_n = \mathcal{G}(X_1, \ldots, X_n) \quad n \geq 1\)

Then \((S_n, n \in \mathbb{N})\) is a sq.r.t. martingale w.r.t. \((\mathcal{F}_n, n \in \mathbb{N})\)

Proof:

(i) \(\mathbb{E}(S_n^2) = n \cdot \mathbb{E}(X_1^2) = n < +\infty \quad \forall n \geq 1\)

(ii) \(S_n\) is \(\mathcal{F}_n\)-measurable: yes, as \(S_n = X_1 + \ldots + X_n\)

(iii) \(\mathbb{E}(S_{n+1} | \mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1} | \mathcal{F}_n) = \mathbb{E}(S_n | \mathcal{F}_n) + \mathbb{E}(X_{n+1} | \mathcal{F}_n)\)

\(= S_n + \mathbb{E}(X_{n+1}) = S_n\) a.s. \(\quad \uparrow\)

\(\mathcal{F}_n\)-measurable \(\Rightarrow X_{n+1} \perp \mathcal{F}_n\)
Remark: If \((\Pi_n, n \in \mathbb{N})\) is a square martingale, then
\[
E(\Pi_{n+1}) = E(E(\Pi_{n+1} | \mathcal{F}_n)) = E(\Pi_n) \quad \forall n \in \mathbb{N}
\]
So \(E(\Pi_n) = E(\Pi_{n-1}) = \ldots = E(\Pi_0) \quad \forall n \in \mathbb{N}
\]
So a martingale is a process with constant expectation; but it is also much more than that!

Ex: The process \((X_n, n \geq 1)\) with \(X_n\) iid r.v. has constant expectation, but it is not a martingale:
\[
E(X_n | \mathcal{F}_n) = E(X_n) = 0 \neq X_n
\]

Illustration for \((S_n, n \in \mathbb{N})\):
Property: Let \((\Pi_n, n \geq 1)\) be a square-integrable martingale.

- \(\mathbb{E}(\Pi_{n+2} | \mathcal{F}_n) = \mathbb{E}(\mathbb{E}(\Pi_{n+2} | \mathcal{F}_{n+1}) | \mathcal{F}_n) = \mathbb{E}(\Pi_{n+1} | \mathcal{F}_n)\)
  
  \[ = \Pi_n \quad \text{for } n, m \in \mathbb{N} \]

Remark: martingale = Markov

- Martingale property: \(\mathbb{E}(\Pi_{n+1} | \mathcal{F}_n) = \Pi_n\)

- Markov property: \(\mathbb{P}(\exists \Pi_{n+1} \in \mathcal{B}^3 | \mathcal{F}_n) = \mathbb{P}(\exists \Pi_{n+1} \in \mathcal{B}^3 | \Pi_n)\)
  
  equivalently:
  
  \[\mathbb{E}(g(\Pi_{n+1}) | \mathcal{F}_n) = \mathbb{E}(g(\Pi_{n+1}) | \Pi_n) \quad \forall g \in C^0(\mathbb{R})\]
Sub- and supermartingales (square-integrable)

Def.: A process \((M_n, n \in \mathbb{N})\) is a sub- or supermartingale w.r.t. a filtration \((\mathcal{F}_n, n \in \mathbb{N})\) if:

1. \(\mathbb{E}(M_n^2) < \infty \quad \forall n \in \mathbb{N}\)

2. \(M_n\) is \(\mathcal{F}_n\)-measurable \(\forall n \in \mathbb{N}\)

3. \(\mathbb{E}(M_{n+1} | \mathcal{F}_n) \geq M_n\quad \text{as} \quad \forall n \in \mathbb{N}\)

Rem.: 1) if \(M\) is a supermartingale, then \(\mathbb{E}(M_{n+1}) \geq \mathbb{E}(M_n)\).

2) Not every process is either a submartingale, or a martingale, or a supermartingale.
Example (simple asymmetric random walk)

\[ S_0 = 0, \ S_n = X_1 + \ldots + X_n \quad \text{with} \quad (X_n, n \geq 1) \ \text{iid r.v.'s with} \]

\[ P(\xi X_i = 1) = p = 1 - P(\xi X_i = -1), \quad 0 < p < 1 \]

\((S_n, n \in \mathbb{N})\) is a \text{sub-}\{ \text{super-}\} martingale if \ \{p \geq \frac{1}{2} \quad p \leq \frac{1}{2} \}

\[ \text{Pf: } \mathbb{E}(S_{n+1} \mid F_n) = \mathbb{E}(S_n \mid F_n) + \mathbb{E}(X_{n+1} \mid F_n) = S_n + \mathbb{E}(X_{n+1}) \geq 0 \ \\
\text{Jensen} \]

Property: If \( M \) is a square-integrable martingale

& \( \varphi : \mathbb{R} \to \mathbb{R} \) is a convex function s.t. \( \mathbb{E}(\varphi(M_n)^2) < \infty \) \( \forall n \in \mathbb{N} \)

Then \((\varphi(M_n), n \in \mathbb{N})\) is a submartingale.

\[ \text{Pf: } \mathbb{E}(\varphi(M_{n+1}) \mid F_n) \geq \varphi(\mathbb{E}(M_{n+1} \mid F_n)) = \varphi(M_n) \]

Jensen
Stopping times & Optional stopping theorem

Def: A random time is a random variable $T: \Omega \to \mathbb{N}$.

Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration on $(\Omega, \mathcal{F}, P)$.

Def: A stopping time w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$ is a random time such that $\{\omega \in \Omega : T(\omega) = n\} \in \mathcal{F}_n$ for $n \in \mathbb{N}$.

Ex: Let $(X_n, n \in \mathbb{N})$ be a discrete-time stochastic process $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ for $n \in \mathbb{N}$.

Then $T_\alpha = \inf\{n \in \mathbb{N} : X_n \geq \alpha\}$ is a stopping time w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$. 
Proof: To be checked: \( \{ T_a = n \} \in \mathcal{F}_n \quad \forall n \in \mathbb{N} \)

\[
\{ T_a = n \} = \left\{ X_k < a \ \forall 0 \leq k \leq n-1 \ \& \ X_n \geq a \right\}
\]

\[
= \left( \bigcap_{0 \leq k \leq n-1} \{ X_k < a \} \right) \cap \{ X_n \geq a \} 
\in \mathcal{F}_n
\]

\[
= \mathcal{F}_k \cap \mathcal{F}_n 
\]

\[
\in \mathcal{F}_n 
\]

\( \checkmark \)
Def: If \((X_n, n \in \mathbb{N})\) is a discrete-time stochastic process and \(T\) is a random time, then we define

\[
X_T(\omega) := X_{T(\omega)}(\omega) = \sum_{n \in \mathbb{N}} X_n(\omega) 1_{\exists T = n}(\omega)
\]

Def: We say that a random time is bounded if there exists \(N \in \mathbb{N}\) such that \(T(\omega) \leq N\) for all \(\omega \in \Omega\) if

\[
(\forall \omega \in \Omega) \quad P(\exists T \leq N) = 1
\]
Optional stopping theorem

1. Let $M$ be a (square-integrable) martingale w.r.t. a filtration $(\mathcal{F}_n, n \in \mathbb{N})$

Let $T$ be a stopping time w.r.t. $(\mathcal{F}_n, n \in \mathbb{N})$ such that $\exists N \in \mathbb{N}$ with $0 \leq T(\omega) \leq N$

Then $\mathbb{E}(\Pi_0) = \mathbb{E}(\Pi_T) = \mathbb{E}(\Pi_N)$

2. If $M$ is a (square-integrable) submartingale, then $(\overset{\mathbb{P}}{\mathbb{E}}(\Pi_0) \leq) \mathbb{E}(\Pi_T) \leq \mathbb{E}(\Pi_N)$

not proven here

\[ \hat{} \]
Proof:

As \( T(\omega) \leq N \), we have

\[
M_T(\omega) = M_{n_0}(\omega) = 2^n \sum_{n=0}^{\infty} E(M_n | \mathcal{F}_n) = E(M_T) = E(M_{n_0}) = 2^n \sum_{n=0}^{\infty} E(M_n | \mathcal{F}_n)
\]

(\text{in case } M \text{ is a submartingale})
When does the optional stopping theorem fail?

- If $T$ is not a stopping time:
  
  $\text{Ex: } T = \text{value of } n \in \mathbb{E}_0..N \text{ such that } M_n \geq M_m \forall m \in \mathbb{E}_0..n$,
  
  Then both $\mathbb{E}(\Pi_T) > \mathbb{E}(\Pi_0)$ & $\mathbb{E}(\Pi_T) > \mathbb{E}(\Pi_N)$.

- If $T$ is not bounded:
  
  $\text{Ex: } \text{Consider } (\xi_n, n \geq 1) \text{ be a sequence of iid r.v.'s st. } P(\xi_1 = +1) = P(\xi_1 = -1) = \frac{1}{2}; \quad \Pi_0 = 0, \quad \Pi_n = \xi_1 + \cdots + \xi_n, n \geq 1$
  
  $M = \text{martingale}, \quad T_a = \text{inf} \{ \exists n \in \mathbb{N}: \Pi_n \geq a \} \quad a \in \mathbb{N}^*$
  
  Then $\mathbb{E}(\Pi_{T_a}) = a > 0 = \mathbb{E}(\Pi_0)$.
Why is it an interesting theorem?

Consider again the previous example with \( \Pi = \) simple symmetric random walk: \( \Pi_n = X_1 + \cdots + X_n \).

\( X\)'s can losses \( \to^+ \frac{1}{\sqrt{n}} \Rightarrow \Pi_n = \) cumulated gain at time \( n \)

\( \forall N \geq 1, \ E(\Pi_N) = E(\Pi_0) = 0 \)

Let \( T = \min\{ \inf \{ n \in N : \Pi_n \geq 10 \} , N \} \)

\( E(\Pi_T) = E(\Pi_0) \)

\( \begin{cases} \geq +10 & \text{with high probability} \\ \leq -\infty & \text{with low probability (when } T=N) \end{cases} \)