Quiz. True or false? (10 points; correct answer = 2pts, wrong answer = -1pt, no answer = 0pt)

a) A stationary distribution satisfying detailed balance is always also a limiting distribution.
□ True   □ False

b) If $X, Y$ are two independent random variables with different distributions, then $\mathbb{P}(X \neq Y) > 0$.
□ True   □ False

c) If a Markov chain is finite, ergodic and reversible, then all the eigenvalues of its transition matrix are strictly less than 1 in absolute value.
□ True   □ False

d) The spectral gap of a finite, ergodic and reversible Markov chain characterizes always completely and precisely its convergence towards equilibrium.
□ True   □ False

e) The stationary distribution of a Metropolis chain necessarily satisfies detailed balance.
□ True   □ False

Exercise 1. (30 points)

Let $(X_n, n \geq 1)$ be a Markov chain with state space $S = \{0, 1\}$, initial distribution $\pi^{(0)}$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad \text{where} \quad 0 < p, q < 1$$

Let $(Y_n, n \geq 1)$ be another Markov chain with same state space $S$ and same transition matrix $P$, but whose initial distribution $\pi$ is also the stationary distribution of the chain.

a) Compute $\pi$.

We consider now the coupled chain $Z = (X, Y)$ with state space $S \times S$ such that $X, Y$ evolve independently according to $P$ as long as $X_n \neq Y_n$, and then evolve together, still according to $P$, as soon as $X_n = Y_n$.

b) Write down the transition matrix $P_Z$ of the chain $Z$.

c) Which states of $Z$ are transient / recurrent?

d) Does the chain $Z$ admit a unique limiting and stationary distribution $\pi_Z$? If yes, compute it; if no, explain why.

e) Express $\mathbb{P}(X_{n+1} \neq Y_{n+1})$ as a function of $\mathbb{P}(X_n \neq Y_n)$.

f) From e), deduce an upper bound on $\max_{i \in S} \|P^n_i - \pi\|_{TV}$.

g) When $p = q$, what value of $0 < p < 1$ leads to the fastest convergence?
Exercise 2. (16 points)
Let $M \geq 1$ be an integer and $N = 4M + 1$. We consider the Markov chain with state space $S = \{0, 1, \ldots, N-1\}$ and transition matrix given by

$$P = \text{circ}(0, 0, 1/2, 0, \ldots, 0, 1/2, 0)$$

Note: In the case $N = 5$, $P = \text{circ}(0, 0, 1/2, 1/2, 0)$.

a) Compute the spectral gap of the chain.

Reminder: If $A$ is an $N \times N$ circulant matrix $A = \text{circ}(c_0, c_1, \ldots, c_{N-1})$, then its eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \exp\left(\frac{2\pi i j k}{N}\right) \quad k = 0, \ldots, N - 1$$

Consider now the modified chain with added self-loops, all of the same weight $0 < \alpha < 1$, and transition probabilities decreased accordingly by a multiplicative factor $1 - \alpha$.

b) For a fixed value of $N$, compute the value of $\alpha$ leading to the largest possible value of the spectral gap for the modified chain.

c) Compute the numerical value of $\alpha$ in the case $N = 5$.

Hints: Use the formula $\cos(2x) = 2\cos^2(x) - 1$ and the fact that $\cos(\pi/5) = \frac{1 + \sqrt{5}}{4}$.

Exercise 3. (16 points)
Let first $i_0 \in \mathbb{Z}$ and $\beta > 0$. Let us then consider the Metropolis chain with state space $S = \mathbb{Z}$ obtained from the base chain with the following transition probabilities:

$$\psi_{ij} = \begin{cases} 
1/2 & \text{if } |j - i| = 1 \\
0 & \text{otherwise}
\end{cases}$$

and with the following acceptance probabilities:

$$a_{ij} = \begin{cases} 
e^{-\beta} & \text{if } (j = i + 1 \text{ and } i \geq i_0) \text{ or } (j = i - 1 \text{ and } i \leq i_0) \\
1 & \text{if } (j = i + 1 \text{ and } i < i_0) \text{ or } (j = i - 1 \text{ and } i > i_0)
\end{cases}$$

a) Compute the limiting distribution $\pi$ of the Metropolis chain.

b) Assuming first that the Metropolis chain is run for a sufficiently long time, what is the probability that state $i_0$ is sampled with this method?

c) For what values of $\beta > 0$ is this probability greater than or equal to $1/2$?

d) Assume now that $i_0 > 0$, that the Metropolis chain starts in position $i = 0$ with probability 1, and that it is run only for $i_0$ steps. What is the probability that state $i_0$ is sampled in this case?