1.a) The chain is finite and irreducible, so it is also positive recurrent. Based on Theorem 1, it has a unique stationary distribution.

Proof of irreducibility: For each pair of states $i$ and $j$ such that $i < j$, we have:
\[ n = j - i : \ p_{ij}^{(n)} \geq p > 0 \]
\[ n = N + M + i - j + 2 : \ p_{ji}^{(n)} \geq p^2 q > 0 \]

1.b) Using basic calculations, it can be shown that the stationary distribution has the form
\[ \pi_i = a \quad \text{if} \quad 0 \leq i < N \]
\[ \pi_i = b \quad \text{if} \quad N < i \leq M \]
\[ \pi_N = a + b \]

At the same time, since $\pi_0 = q \pi_N$ we have $a = \frac{q}{p} b$. And as a result
\[ \sum_{i=0}^{N+M} \pi_i = (N+1) a + (M+1) b = \left( N + 1 + (M + 1) \frac{p}{q} \right) a = 1 \]
and
\[ a = \frac{q}{(M + 1) p + (N + 1) q} \quad b = \frac{p}{(M + 1) p + (N + 1) q} \]

1.c) Whenever $M > 1$ or $N > 1$, the detailed balance equation is obviously not satisfied (because of the irreversible deterministic transitions on either the first $N$ states or the last $M$ states). Using the stationary distribution we found in the previous part, and for the case that $N = M = 1$, we have
\[ \pi_0 = \frac{q}{2}, \quad \pi_1 = \frac{1}{2}, \quad \pi_2 = \frac{p}{2} \]

It is obvious that the detailed balance equation is satisfied for all values of $p, q > 0$ and $p + q = 1$.

1.d) For the state $N$, we have
\[ \{ k : p_{NN}^{(k)} > 0 \} = \{ k_1 (N + 1) + k_2 (M + 1) : k_1, k_2 \in \mathbb{N} \} \]
and hence $d_N = \gcd\{N + 1, M + 1\}$. The minimal condition for aperiodicity: $M + 1$ and $N + 1$ should be relatively prime.

1.e) Since the chain is ergodic, we have
\[ \lim_{n \to \infty} \mathbb{P}(X_n < N) = \lim_{n \to \infty} \sum_{i=0}^{N-1} \pi_i^{(n)} = \sum_{i=0}^{N-1} \pi_i = \frac{Nq}{(M + 1) p + (N + 1) q} \]

1.f) Setting $\lim_{n \to \infty} \mathbb{P}(X_n < N) = \lim_{n \to \infty} \mathbb{P}(X_n > N)$, we obtain:
\[ \frac{Nq}{(M + 1) p + (N + 1) q} = \frac{Mp}{(M + 1) p + (N + 1) q} \quad \text{so} \quad Nq = Mp \]
2.a) The chain is irreducible and has a stationary distribution, so based on Theorem 1, it is also positive recurrent.

Proof of irreducibility: For each pair of states $i$ and $j$ such that $i < j$, we have:

$$n = j - i : \quad p^{(n)}_{ij} = p^{j-i} > 0$$

$$n = j - i : \quad p^{(n)}_{ji} = q^{j-i} > 0$$

Finding the stationary distribution: Since the matrix $P$ is tridiagonal, we find $\pi$ by solving the detailed balance equation:

$$\pi_{i+1} = \left(\frac{p}{q}\right) \pi_i \quad \text{so} \quad \pi_i = \left(\frac{p}{q}\right)^i \pi_0$$

and using $\sum_{i\geq0} \pi_i = 1$, so $\pi_0 = \frac{q-p}{q}$.

2.b) Using Theorem 1 and the result of part a:

$$\mu_i = \pi_i^{-1} = \frac{q}{q-p} \left(\frac{p}{q}\right)^i$$

BONUS 2.c) Since the chain is ergodic, and defining $\alpha = \frac{p}{q}$, we have

$$\lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \sum_{k=0}^{\infty} k \pi_k^{(n)} = \sum_{k=0}^{\infty} k \pi_k = \frac{q}{q-p} \sum_{k=0}^{\infty} k \alpha^k = \frac{q}{q-p} \alpha \sum_{k=0}^{\infty} \frac{1}{\alpha^k} = \frac{q}{q-p} \frac{\alpha}{1-\alpha} = \frac{p}{q-p}$$

3.a) To have the detailed balance equation satisfied, we should have

$$p \pi_0 = \frac{1}{2} \pi_1 \quad (1-q) \pi_2 = \frac{1}{2} \pi_1 \quad (1-p) \pi_0 = q \pi_2$$

by removing $\pi_1$ from the equations we have

$$p \pi_0 = (1-q) \pi_2 \quad (1-p) \pi_0 = q \pi_2$$

which is satisfied if and only if $\frac{pq}{(1-p)(1-q)} = 1$, i.e., $p + q = 1$. Then we have

$$\pi_0 = \pi_2 = \frac{1}{2 + 2p} \quad \pi_1 = \frac{2p}{2 + 2p}$$

3.b) Considering $\lambda_0 = 1$, we have

$$\lambda_1 + \lambda_2 = \text{Tr}(P) - 1 = -1 \quad \lambda_1 \lambda_2 = \det(P) = pq$$

Then we have

$$\lambda_1^2 + \lambda_1 + pq = \lambda_1^2 + (p + q) \lambda_1 + pq = (\lambda_1 + p)(\lambda_1 + q) = 0$$

Therefore, eigenvalues of $P$ are $\{1, -p, -q\}$.

3.c) $\gamma = \min\{p, q\}$. 

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