1. **a)** The transition probabilities are given by the following formula:

\[
p(x, y) = \begin{cases} 
  \frac{\pi(x_u' | x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d}, & \text{if } y = (x_1, \cdots, x_{u-1}, x_u', x_{u+1}, \cdots, x_d) \\
  \frac{\sum_{u=1}^{d} \pi(x_u | x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d}, & \text{if } y = x, \\
  0, & \text{otherwise.}
\end{cases}
\]

(1)

**b)** Let us now show that the detailed balance equation is satisfied.

First note from (1) that \( p(x, y) = 0 \) if and only if \( x \) and \( y \) differ in more than a single component. Hence, \( p(x, y) = 0 \) if and only if \( p(y, x) = 0 \) and the detailed balance equation is satisfied since both sides are zero.

Then, suppose that \( x \) and \( y \) differ in component \( u \), i.e., \( x = (x_1, \cdots, x_{u-1}, x_u, x_{u+1}, \cdots, x_d) \) and \( y = (x_1, \cdots, x_{u-1}, x_u', x_{u+1}, \cdots, x_d) \). Therefore, from (1) we have

\[
\pi(x)p(x, y) = \frac{\pi(x)\pi(x_u' | x_1, \cdots, x_{u-1}, x_{u+1}, \cdots, x_d)}{d} = \frac{g(x)g(y)}{Z \cdot \sum_{y_u \in S} g(x_1, \cdots, x_{u-1}, y_u, x_{u+1}, \cdots, x_d)} = \pi(y)p(y, x).
\]

As a result, this algorithm can be viewed as a Metropolis-Hastings algorithm where the base chain has the transition probabilities \( p(x, y) \) defined in (1) and the acceptance probability \( a(x, y) = 1 \) for any \( x, y \in S^d \). In words, every move is always accepted. Note that every base chain which satisfies the detailed balance equation induces a Metropolis Hastings algorithm in which every move is always accepted. To see this, consider the base chain \( \psi_{ij} \) s.t. \( \pi_j \psi_{ij} = \pi_j \psi_{ji} \) for any \( i, j \) in some state space \( S \). Then, as the base chain is not necessarily symmetric, the acceptance probability is given by

\[
a_{ij} = \min \left( 1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}} \right) = 1.
\]
2. a) The transition probabilities are given by
\[
\begin{align*}
p_{01} &= \psi_{01} \min \left( 1, \frac{\psi_{01} \pi_0}{\psi_{01} \pi_0} \right) = \frac{e^{-2\beta}}{2} \\
p_{10} &= \psi_{10} \min \left( 1, \frac{\psi_{10} \pi_0}{\psi_{10} \pi_1} \right) = \frac{1}{2} \\
p_{02} &= p_{20} = p_{11} = 0 \\
p_{12} &= \psi_{12} \min \left( 1, \frac{\psi_{12} \pi_2}{\psi_{12} \pi_1} \right) = \frac{1}{2} \\
p_{21} &= \psi_{21} \min \left( 1, \frac{\psi_{21} \pi_2}{\psi_{21} \pi_1} \right) = \frac{e^{-\beta}}{2} \\
p_{22} &= 1 - \frac{e^{-\beta}}{2} \\
p_{00} &= 1 - e^{-\frac{3\beta}{2}} \\
p_{11} &= 1 - e^{-\frac{3\beta}{2}} \\
p_{20} &= 1 - e^{-\frac{3\beta}{2}} \\
p_{21} &= 1 - e^{-\frac{3\beta}{2}} \\
p_{22} &= 1 - e^{-\frac{3\beta}{2}}.
\end{align*}
\] (2)

b) Let us now check that the detailed balance equation is satisfied:
\[
\begin{align*}
p_{01} \pi_0 &= \frac{1}{2} e^{-2\beta} = p_{10} \pi_1 \\
p_{02} \pi_0 &= 0 = p_{20} \pi_2 \\
p_{12} \pi_1 &= \frac{1}{2} e^{-2\beta} = p_{21} \pi_2.
\end{align*}
\]

c) As usual, there are several methods to compute the eigenvalues. For example, one can find the three solutions \(\lambda_0, \lambda_1, \lambda_2\) to the equation
\[
\det(P - \lambda I) = 0,
\] (3)
where \(I\) is the \(3 \times 3\) identity matrix and \(P\) the matrix of the transition probabilities computed in (2).

Another (perhaps even simpler method) method is to solve the following system of equations:
\[
\begin{align*}
\lambda_0 &= 1 \\
\lambda_0 + \lambda_1 + \lambda_2 &= \text{tr}(P), \\
\lambda_0 \cdot \lambda_1 \cdot \lambda_2 &= \det(P)
\end{align*}
\]
as we know that the largest eigenvalue is 1, the sum of the eigenvalues equals the trace of \(P\), and their product equals the determinant of \(P\).

Consequently, we obtain
\[
\begin{align*}
\lambda_0 &= 1 \\
\lambda_1 &= -\frac{e^{-2\beta}}{4} - \frac{e^{-\beta}}{4} + \frac{1}{2} + \frac{1}{4} \sqrt{e^{-4\beta} - 2e^{-3\beta} + e^{-2\beta} + 4}, \\
\lambda_2 &= -\frac{e^{-2\beta}}{4} - \frac{e^{-\beta}}{4} + \frac{1}{2} - \frac{1}{4} \sqrt{e^{-4\beta} - 2e^{-3\beta} + e^{-2\beta} + 4}
\end{align*}
\]

d) The spectral gap is given by
\[
\gamma = 1 - \lambda_1 = \frac{1}{2} + \frac{e^{-2\beta}}{4} + \frac{e^{-\beta}}{4} - \frac{1}{4} \sqrt{e^{-4\beta} - 2e^{-3\beta} + e^{-2\beta} + 4}.
\] (4)

Therefore, when \(\beta\) is large, we have
\[
\gamma \approx \frac{1}{4} e^{-\beta}.
\] (5)

Remark. The value of \(\beta\) has to be tuned carefully and there is an inherent trade-off in its choice. If we pick \(\beta\) too large, then the spectral gap is small and the convergence to the global minimum occurs very slowly. On the other hand, if we pick \(\beta\) too small, we might not be able to visit all the states and, therefore, we might get stuck in the local minimum (=state 2).