Approval Voting

STEVEN J. BRAMS
New York University

PETER C. FISHBURN
Pennsylvania State University

Approval voting is a method of voting in which voters can vote for ("approve of") as many candidates as they wish in an election. This article analyzes properties of this method and compares it with other single-ballot nonranked voting systems. Among the theorems proved is that approval voting is the most sincere and most strategyproof of all such voting systems; in addition, it is the only system that ensures the choice of a Condorcet majority candidate if the preferences of voters are dichotomous. Its probable empirical effects would be to (1) increase voter turnout, (2) increase the likelihood of a majority winner in plurality contests and thereby both obviate the need for runoff elections and reinforce the legitimacy of first-ballot outcomes, and (3) help centrist candidates, without at the same time denying voters the opportunity to express their support for more extremist candidates. The latter effect's institutional impact may be to weaken the two-party system yet preserve middle-of-the-road public policies of which most voters approve.

1. Introduction

In a democratic election between two candidates, election by simple majority is probably the most common voting procedure in use today and has, in addition, certain desirable features (May, 1952; Rae, 1969; Straffin, 1977). When there are three or more candidates, however, and only one is to be elected to a specified office or position, there are widespread differences of opinion as to how the winner should be determined.

These differences are perhaps best epitomized by Condorcet's criticism (1785) of Borda's method (1781), which is recounted along with later proposals in Black (1958). Borda recommended that in an election among three candidates the winner be the candidate with the greatest point total when awards of two points, one point and zero points are made, respectively, to each voter's most-preferred, second most-preferred and least-preferred candidate. Condorcet argued to the contrary that the winner ought to be the candidate who is preferred by a simple majority of voters to each of the other candidates, provided that such a majority candidate exists, and showed that Borda's method can elect a candidate other than the majority candidate. Although a number of writers have accepted Condorcet's criterion, it leaves open the question of which candidate should win when there is no majority candidate (the "paradox of voting"). Fishburn (1977b) reviews a number of methods for determining a winner from voters' preferences when there is no majority candidate and concludes that, although some methods are better than others, there is no obviously best method.

Probably the two most common voting systems used in multicandidate elections today are single and double plurality systems. Under single plurality, each voter can vote for one candidate, and the candidate with the greatest vote total wins the election. Double plurality is a two-ballot system. Its first ballot is the same as the single plurality ballot; the second or runoff ballot is a simple majority ballot involving the two candidates with the largest vote totals from the first ballot. There are of course many other systems that have been or could be used, and we shall mention some of these as we proceed.

A primary purpose of this article is to examine a relatively simple but rarely used type of voting system with several very attractive characteristics. The designation used here for this type of system is *approval voting*. Under approval voting, voters are allowed to vote for ("approve of") as many candidates as they wish but cannot cast more than one vote for each candidate, as under cumulative voting (Brams, 1975, Ch. 3). Voters are not asked to rank their chosen candidates. The winner is the candidate with the greatest vote total.

As in any other system, special provisions must be made for tied outcomes under approval voting, a situation which will be handled here in a general way without specifying an explicit tie-breaking procedure. Although the basic idea of approval voting can be extended to two-ballot systems (Brams and Fishburn, 1978), we shall concentrate on the single-ballot system.

One of the potentially attractive features of approval voting is that it allows voters the maximum

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number of choices in a single-ballot election in which they can vote for, but not rank, the candidates. If there are $m \geq 3$ candidates, under approval voting a voter can cast either an approval vote or no vote for each candidate, giving each voter 2^m possible voting strategies. However, since an abstention (vote for none) has the same net effect as a vote for all candidates, the number of effectively different choices is $2^m - 1$. By contrast, the single plurality system allows $m+1$ different choices (a vote for one of the m candidates or an abstention), and various other single-ballot systems generally allow between $m+1$ and $2^m - 1$ different strategies if candidates are not ranked by voters.

The effects on voters and potential outcomes of approval voting versus single plurality is vividly illustrated in the 1970 New York State race for United States Senate among James R. Buckley, Charles E. Goodell, and Richard L. Ottinger. In this election, the conservative candidate, Buckley, was elected with 39 percent of the vote under the single plurality system, though either of the more liberal candidates, Goodell or Ottinger, was almost surely preferred by a majority of voters. If approval voting had been used, the outcome probably would have been different since a number of the 61 percent who supported either Goodell or Ottinger (24 percent voted for Goodell, 37 percent for Ottinger) would have voted for both Goodell and Ottinger in an attempt to ensure that Buckley would not win. (In fact, Stratmann [1977] estimated, based on work reported in Stratmann [1974], that under approval voting Goodell would have won with about 59 percent of the vote to about 55 percent each for Buckley and Ottinger; whereas Buckley and Ottinger would have received significant support from Goodell voters, Goodell would have benefited from the support of *both* Buckley and Ottinger supporters.) It may also be noted that Buckley would almost surely have lost under a two-ballot plurality system.

In this article we shall analyze more generally the points suggested by this example. It will be assumed that each voter has a definite preference order on the candidates so that, for any two candidates, a voter prefers one to the other or is indifferent between them (connectivity); and a voter's preference and indifference relations on all candidates are transitive (transitivity). Besides connectivity and transitivity, we will assume that nonabstaining voters use admissible strategies, as defined in section 2, which does *not* imply that they will always vote for their most-preferred candidates. Admissible strategies will also be identified for other types of voting systems and compared to those for approval voting.

In a sense specified precisely in section 3, we shall prove that approval voting is both more sincere and more strategyproof than every other single-ballot voting system that does not ask voters to rank candidates. In other words, among single-ballot systems, approval voting is most likely to encourage voters to report their true preferences, and it is most likely to offer the voter a unique admissible voting strategy.

In section 4 we examine Condorcet's criterion in the special but interesting case of dichotomous preferences. Dichotomous preferences obtain when a voter is able to divide the candidates into a more-preferred subset and a less-preferred subset in such a way as to be indifferent among the candidates within each subset. The main results of the section are that if all voters have dichotomous preferences, then (1) the outcome of a single-ballot election using approval voting will be a Condorcet majority candidate, and (2) approval voting is the only single-ballot nonranked voting system that has this property. If preferences are not dichotomous, the picture is less clear.

In section 5 we consider the possible effects of approval voting on presidential elections. In particular, we focus on the 1968 multicandidate race and report on some estimates, based on survey responses, of the likely outcome had there been approval voting in this election. We discuss probable empirical effects of approval voting and conclude with some normative observations.

2. Admissible Strategies

We begin this section by discussing certain aspects of voters' preferences and then develop a notion of dominance between voting strategies for single-ballot nonranked voting systems. Next we describe the set of all such systems and define the concept of an admissible voting strategy for a given system and a given voter's preference order on the candidates. We then characterize the set of admissible voting strategies for every system and every preference order on the candidates. This is followed by a discussion of admissible strategies for approval voting and a comparison between admissible approval voting strategies and admissible strategies for two other interesting voting systems. Additional comparisons concerning sincere voting strategies and strategyproof voting systems are presented in section 3.

Voters' Preferences. Throughout this article, individual candidates will be denoted as $a, b, c, \cdots$, and subsets of candidates will be denoted as $A, B, C, A_i, \cdots$. For any two subsets A and B, $A \cup B = \{a: a \in A \text{ or } a \in B\}$, the union of A and B, and $A \backslash B = \{a: a \in A \text{ and } a \notin B\}$, the set of all candidates who are in A and not in B. In addition, $\{a\}$ is the subset of candidates that contains only candidate a, $\{a, b\}$ is the subset consisting of candidates a and b, and so forth.

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A voter's strict preference relation on the candidates will be denoted as P, so that aPb means that the voter definitely prefers a to b. Similarly, R will denote a voter's nonstrict preference relation on the candidates, so that aRb means that the voter likes a as much as b. Alternatively, aRb means that the voter either strictly prefers a to b or is indifferent between a and b. According to the connectivity and transitivity assumptions noted earlier, the set of all candidates can be partitioned into nonempty subsets, say $A_1, A_2, \cdots, A_n$, for a given P so that the voter is indifferent among all candidates within each A_i and strictly prefers every candidate in A_i to every candidate in A_j if and only if $i<j$. According to this designation, A_1 is the voter's *subset of most preferred candidates* and A_n is the voter's *subset of least preferred candidates.* If the voter is indifferent among all candidates, then A_n and A_1 are the same, but otherwise A_1 and A_n are disjoint. The following comprehensive definition introduces a number of terms that will be used in this and later sections.

DEFINITION 1. Suppose P partitions the set of all candidates into $n \geq 1$ nonempty subsets $A_1, A_2, \cdots, A_n$ so that the voter is indifferent among all candidates within each A_i and has aPb when $a \in A_i$ and $b \in A_j$ if, and only if, $i<j$. Then P is *unconcerned* if and only if $n=1$; P is *dichotomous* if and only if $n=2$; P is *trichotomous* if and only if $n=3$; and P is *multichotomous* if and only if $n \geq 4$. In addition, a subset of candidates B is *high for P* if and only if whenever it contains a candidate in A_j it contains all candidates in A_i for every $i<j$; and B is *low for P* if and only if whenever it contains a candidate in A_i it contains all candidates in A_j for every $j>i$.

A voter who has an unconcerned P will be referred to as an unconcerned voter since s/he is indifferent among all candidates. If P is unconcerned, then every subset of candidates is both high and low for P.

When P is concerned—that is, when P is either dichotomous, trichotomous, or multichotomous—exactly two subsets of candidates are both high and low for P, namely the empty set and the set of all candidates. If P is trichotomous on a set of five candidates $\{a, b, c, d, e\}$, say with $A_1=\{a\}$, $A_2=\{b, c\}$ and $A_3=\{d, e\}$, then P has six high subsets in addition to the two just mentioned, namely $\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,c,d\}$ and $\{a,b,c,e\}$; and P has six low subsets in addition to the empty set and the whole set, namely $\{e\}$, $\{d\}$, $\{e, d\}$, $\{c, d, e\}$, $\{b, d, e\}$ and $\{b, c, d, e\}$. In all cases, the number of high subsets is equal to the number of low subsets since a subset is high if and only if its complement (all other candidates) is low.

The characterizations of dominance and admissible strategies to be developed momentarily depend not only on the relations P and R as applied to individual candidates but also on extensions of these relations to subsets of candidates. The reason for this is that we view dominance and admissibility to be based on individual preferences between potential outcomes of a vote, where the *outcome* of a given vote in the single-ballot nonranked context is the set of all candidates who have the greatest vote total from the single ballot. In most real cases there will be no ties and hence the outcome will consist of the one candidate with the largest vote total. However, if ties occur for the largest total, then the outcome will be a subset of two or more candidates.

Instead of specifying a method for determining an ultimate winner when the outcome of a vote contains two or more candidates, we shall proceed on the basis of assumptions that relate preferences between potential vote outcomes to preferences on the individual candidates. The symbols P and R that are used for a voter's preferences between individual candidates will also be used for the voter's preferences between subsets of candidates viewed as potential vote outcomes. Thus, APB means that the voter prefers outcome A to outcome B, and ARB means that s/he finds A as preferable as B.

When A and B are one-candidate subsets, say $A=\{a\}$ and $B=\{b\}$, we shall naturally assume that APB if and only if aPb, and that ARB if and only if aRb. It is assumed also that, for any nonempty A and B, APB and BRA cannot both hold. In addition, the following will be assumed for all candidates a and b and for all subsets of candidates A, B and C:

ASSUMPTION P. If aPb then $\{a\}P\{a, b\}$ and $\{a, b\}P\{b\}$.

ASSUMPTION R. If $A \cup B$ and $B \cup C$ are not empty and if aRb, bRc and aRc for all $a \in A$, $b \in B$ and $c \in C$, then $(A \cup B)R(B \cup C)$.

Assumption P asserts that if candidate a is preferred to candidate b, then outcome $\{a\}$ is preferred to the tied outcome $\{a, b\}$, which in turn is preferred to $\{b\}$. This seems quite reasonable, regardless of how the tie between a and b might be broken when $\{a, b\}$ occurs, if the voter believes that a and b each has a positive probability of being elected when the two are tied after the initial ballot. This will surely be the case if ties are resolved probabilistically (by "coin flips") on the candidates in the outcome, but we expect that it will be true also for most other tie-breaking procedures.

Assumption R says that if everything in A is at least as good as everything in B and C, and if everything in B is at least as good as everything in C, then outcome $A \cup B$ will be as good as outcome $B \cup C$. If ties are broken randomly, then the conclusion of Assumption R says that the random choice of a winner from the union of A and B

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is as good as the random choice of a winner from the union of B and C.

Although Assumption R can be expected to hold in other cases, it is possible to imagine situations that challenge its credibility. For example, suppose $\{a, b, c, d\}$ is the set of candidates, and aPb, bPc, and cPd. Suppose further that there are two ballots, and the lowest candidate is eliminated on the first ballot. If there is a tie between a and d for the lowest vote total, then either $\{a, b, c\}$ or $\{b, c, d\}$ will be in the runoff. Assume that $A \cup B = \{a, b, c\}$ and $B \cup C = \{b, c, d\}$. If the voter with preference order P is convinced that c will be elected in a runoff among a, b and c and that b will be elected in a runoff among b, c and d, then we would expect this voter to prefer $B \cup C$ to $A \cup B$ in contradiction to Assumption R. Despite this possibility, Assumption R seems plausible in most situations and will be used in our analysis.

We note in passing that Assumption R implies that an unconcerned voter will be indifferent among all outcomes as well as among all individual candidates. It then follows from the definition of dominance developed below that an unconcerned voter never has any dominated strategies.

Dominance Between Strategies. In the general context of single-ballot nonranked voting systems, a *strategy* is any subset of candidates. For identification purposes, we shall usually use S and T rather than $A, B, C, \cdots$ to identify strategies. A voter uses strategy S if s/he votes for each candidate in S and no candidate not in S.

A strategy is *feasible* for a particular voting system if and only if it is permitted by that system. We shall assume that the abstention strategy—which is the empty subset of candidates—is always feasible. For every other strategy a voter's ballot is counted if and only if s/he uses a feasible strategy. In addition to abstention, single plurality has m feasible strategies when there are m candidates. Under approval voting, all strategies are feasible.

Our central notion of admissible strategies depends on feasibility and on dominance. Roughly speaking, strategy S dominates strategy T for a particular voter if s/he likes the outcome of S as much as the outcome of T in every possible circumstance and strictly prefers the outcome of S to the outcome of T in at least one circumstance. To define dominance precisely, we first define a *contingency* as a function f that assigns a nonnegative integer to each candidate. A contingency is interpreted as specifying the numbers of votes each candidate receives from all voters *other than* the voter for whom dominance is being defined. Given a contingency f and a strategy S for our focal voter, we shall let $F(S, f)$ denote the outcome of the vote. That is, $F(S, f)$ is the subset of candidates who have the greatest vote total under f and S. For any candidate a and strategy S let $S(a) = 1$ if $a \in S$ with $S(a) = 0$ otherwise. Then, with $f(a)$ the integer assigned by contingency f to candidate a, $a \in F(S, f)$ if and only if $f(a) + S(a) \geq f(b) + S(b)$ for all candidates $b \neq a$. That is, a necessary and sufficient condition for candidate a to be contained in the outcome is that s/he receive at least as many votes from *all* voters as does every other candidate.

One of the main tasks of our analysis is to determine strategies for a voter that lead to outcomes s/he most prefers. Although different strategies may be preferred under different contingencies, some strategies are uniformly as good as or better than other strategies regardless of the contingency. That is, one strategy may dominate another strategy.

DEFINITION 2. Given the strict and nonstrict preference relations P and R for a voter, strategy S *dominates* strategy T, or S dom T for this voter, if and only if $F(S, f)RF(T, f)$ for all possible contingencies f and $F(S, f)PF(T, f)$ for at least one contingency.

This definition does not require S and T to be feasible strategies and it is therefore applicable to all single-ballot nonranked voting systems. Feasibility will enter our analysis explicitly through the definition of admissibility that will follow shortly.

Because of the $F(S, f)PF(T, f)$ requirement at the end of Definition 2, it follows from an earlier remark about Assumption R that no strategy is dominated if P is unconcerned. The following theorem characterizes dominance between strategies for all concerned P. The definitions of high and low subsets are given in Definition 1, and $S \backslash T$ is the set of all candidates that are in S and not in T.

THEOREM 1 (Dominance). *Suppose* P *is concerned and Assumptions* P *and* R *hold. Then* S *dom* T *for* P *if and only if* $S \neq T$, $S \backslash T$ *is high for* P, $T \backslash S$ *is low for* P, *and neither* $S \backslash T$ *nor* $T \backslash S$ *is the set of all candidates.*

Theorem 1 and other theorems are proved in the Appendix.

Although Theorem 1 is predicated on Assumptions P and R, the necessary and sufficient conditions for S dom T do not explicitly use the P and R relations on the outcomes. That is, *dominance between strategies can be determined completely on the basis of the voter's strict preference relation* P *on the individual candidates.* This greatly simplifies the identification of dominated strategies for a voter.

For example, if the set of candidates is $\{a, b, c\}$, and P is trichotomous with a preferred to b and b preferred to c, then Theorem 1 says that strategy $\{a\}$, under which the voter votes only for his or her most-preferred candidate, dominates strategies

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$\{c\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ and the abstention strategy. Moreover, these are the only strategies that $\{a\}$ dominates.

Theorem 1 also tells us that $\{a\}$ does not dominate $\{b\}$ since $\{b\}\backslash\{a\}=\{b\}$ is not low for P. However, $\{b\}$ is dominated by $\{a, b\}$ according to Theorem 1 since $S \neq T$, $\{a, b\}\backslash\{b\}=\{a\}$ is high for P, $\{b\}\backslash\{a, b\}=\varnothing$ (the empty set, or abstention) is low for P, and neither $\{a\}$ nor $\varnothing$ is the set of all candidates.

Under approval voting, this says that if voters consider voting for their second choice b, then they should also vote for their first choice a since the latter strategy is as good as, and sometimes better than, the strategy of voting for b alone. However, under single plurality, a vote for b alone could be a voter's best strategy since in this case $\{b\}$ is not dominated by any other feasible strategy. As we shall show below (see Definition 4), strategy $\{b\}$ is "admissible" for single plurality voting but "inadmissible" for approval voting.

Admissible Strategies. We shall shortly present a theorem that characterizes all admissible strategies for every concerned P and for all single-ballot voting systems that do not ask voters to rank candidates. This will be followed by comments on admissible strategies for approval voting and for two other simple voting systems.

First, however, we need some definitions. Single-ballot nonranked voting systems will be identified by the numbers of candidates that voters are allowed to vote for if they want their ballots to count.

DEFINITION 3. Suppose there are m candidates. Then a single-ballot nonranked *voting system* is a nonempty subset s of $\{1, 2, \cdots, m-1\}$.

As noted earlier, abstention will be considered feasible for all voting systems. Since a vote for all candidates is tantamount to an abstention so far as the determination of the outcome of a vote is concerned, we have not included m as a possible number in s in Definition 3. This will not present problems with our analysis of admissibility since, when it is allowed, a vote for all m candidates (like an abstention) is dominated by some other feasible strategy whenever P is concerned.

According to Definition 3 and the sentence preceding it, the single plurality system is system $\{1\}$. In other words, under single plurality, a voter is allowed to vote for only one candidate. Under system $\{2\}$, each voter must vote for exactly two candidates if s/he wants his vote to count. System $\{1, 2\}$ allows a voter to vote for either one or two candidates. System $\{1, 2, \cdots, m-1\}$ is the approval voting system. Every system s that omits one or more integers in $\{1, 2, \cdots, m-1\}$ is of course different from the approval voting system. As we shall demonstrate in the next two sections, approval voting is superior to all these other systems in several important respects.

For any subset A of candidates, let $|A|$ denote the *number* of candidates in A. Then strategy S is feasible for system s if and only if either S is the abstention strategy or $|S| \in s$. This characterization of feasibility is consistent with Definition 3 and our previous use of the term. It is now combined with the dominance relation to provide a formal definition of admissibility.

DEFINITION 4. Strategy S is *admissible* for system s and preference order P if and only if S is feasible for s and there is no strategy T that is also feasible for s and has T dom S for P.

As noted in the introduction, our analysis is based on the assumption that nonabstaining voters use admissible strategies. Although a certain amount of formal development has been needed to give a precise definition of admissibility, the intuitive sense of admissibility is easily understood. In effect, Definition 4 says that a voting strategy is admissible for a voter within the context of a specific single-ballot nonranked voting system if and only if (i) that strategy is permitted by the system, and (ii) there is no other permissible strategy—for all possible ways in which other voters may vote—that yields an outcome that is as good for our voter, and is better in at least one instance, as the admissible strategy.

Definition 4 suggests that, because of the feasibility requirement, a strategy feasible for each of two voting systems may be admissible for one system but inadmissible for the other. Indeed, in our earlier example in which $aPbPc$, we noted that strategy $\{b\}$ is admissible for single plurality but inadmissible for approval voting.

Before examining these specific systems in more detail, we state a theorem that characterizes all admissible strategies for every voting system and every concerned preference order. In the statement of the theorem and in later discussion we shall let

$M(P)=A_1$, the subset of most-preferred candidates under P,
$L(P)=A_n$, the subset of least-preferred candidates under P,

where A_1 and A_n are as given in Definition 1.

THEOREM 2 (Admissibility). *Suppose* P *is concerned and Assumptions* P *and* R *hold. Then strategy* S *is admissible for system* s *and preference order* P *if and only if* S *is feasible for* s *and either* C1 *or* C2 *(or both) holds:*

C1 : *Every candidate in* M(P) *is in* S, *and it is impossible to divide* S *into two nonempty subsets* S_1 *and* S_2 *such that* S_1 *is feasible for* s *and* S_2 *is low for* P;

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C2: *No candidate in* L(P) *is in* S, *and there is no nonempty subset* A *of candidates disjoint from* S *such that* A∪S *is feasible for* s *and* A *is high for* P.

Since the abstention strategy satisfies neither $C1$ nor $C2$, it is never admissible for a concerned voter. Because abstention is inadmissible in our formal sense, a vote for all m candidates must likewise be inadmissible if it is permitted. Thus, though we assume that the abstention and "vote for all" strategies are feasible, we omit them from the formal analysis since they are always inadmissible for a concerned voter.

The criteria of Theorem 2 can be applied to determine all admissible strategies for a given s and concerned P. To illustrate, suppose $m=5$ and $s=\{1, 3\}$, which is a rather unorthodox system that allows a voter to vote for either one or three candidates. Given P defined by $A_1=M(P)=\{a\}$, $A_2=\{b,c\}, A_3=\{d\}$ and $A_4=L(P)=\{e\}$, the reader can verify that

strategies $\{a\}$, $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are admissible both by criterion $C1$ and by criterion $C2$;
strategies $\{a, b, e\}$ and $\{a, c, e\}$ are admissible by $C1$ only;
strategy $\{b, c, d\}$ is admissible by $C2$ only;

and no other feasible strategy is admissible. For example, strategy $\{b\}$ fails $C1$ since it excludes a, and it fails $C2$ since $A\cup\{b\}$ is feasible for s and $A=\{a, c\}$ is high for P: in other words, feasible $\{a, b, c\}$ dominates feasible $\{b\}$.

Theorem 2 provides general criteria of admissibility for all single-ballot nonranked voting systems. Although these criteria do not have a simple interpretation in the general case, they can be rendered much more perspicuous in particular cases, as we shall next show.

Admissible Strategies for Approval Voting. When Theorem 2 is applied to approval voting, we obtain the following result.

COROLLARY 1. *Strategy* S *is admissible for approval voting and a concerned* P *if, and only if,* S *contains all candidates in* M(P) *and none in* L(P).

Hence concerned voters use one of their admissible strategies under approval voting if and only if they vote for every one of their most-preferred candidates and do not vote for any of their least-preferred candidates. Thus, if $m=4$ and a voter prefers a to b to c to d, then his admissible strategies are $\{a\}$, $\{a, b\}$, $\{a, c\}$ and $\{a, b, c\}$. To illustrate a case in which this voter may consider $\{a, c\}$—consisting of the voter's first and third choices—the best admissible strategy, we first state a result of Brams (1976, 1978a, 1978c) that follows directly from Corollary 1.

COROLLARY 2. *A voter has a unique admissible strategy under approval voting if and only if his or her preference order* P *is dichotomous. This unique strategy is the voter's subset of most-preferred candidates.*

Now consider again a voter with preference order *abcd* (i.e., s/he prefers a to b to c to d) and suppose that all other voters have dichotomous preferences with a indifferent to b and c indifferent to d. Some of the others prefer a and b to c and d, which we write as $(ab)(cd)$—with parentheses denoting indifference subsets—while the rest prefer c and d to a and b, or $(cd)(ab)$. Suppose further that all other voters use their unique admissible strategies, so that $f(a)=f(b)$ and $f(c)=f(d)$ for whatever contingency obtains. Now assume that the voter with preference order *abcd* estimates that the difference between $f(a)$ and $f(c)$ is likely to be more than one vote. Then, assuming that $f(a)>f(c)+1$ and $f(c)>f(a)+1$ are each thought to be fairly likely, $\{a, c\}$ will probably be the best strategy for our voter since $f(a)>f(c)+1$ implies that

$$F(\{a, c\}, f) = F(\{a\}, f) = \{a\} \quad \text{and}$$
$$F(\{a, b\}, f) = F(\{a, b, c\}, f) = \{a, b\},$$

and $f(c)>f(a)+1$ implies that

$$F(\{a, c\}, f) = F(\{a, b, c\}, f) = \{c\} \quad \text{and}$$
$$F(\{a\}, f) = F(\{a, b\}, f) = \{c, d\}.$$

Hence $\{a, c\}$ ensures (1) the election of our voter's most-preferred candidate when $f(a)>f(c)+1$ and (2) the defeat of our voter's least-preferred candidate when $f(c)>f(a)+1$.

Comparisons with Other Systems. Because approval voting offers more feasible strategies than single plurality and the other systems identified in Definition 3, it might appear that it will (i) confuse voters by its large number of options and (ii) be more liable to strategic manipulation than other systems. Not only is (ii) categorically false, as we shall show in section 3, but (i) is not justified either, as we shall next show.

The basis of our arguments is the assumption that concerned voters who do not abstain will entertain only admissible strategies, hence that comparisons between different voting systems should depend only on *admissible* strategies for those systems. Thus the question of the number of options should be based on the number of admissible strategies and not on the number of feasible strategies.

We will compare two other systems with approval voting to illustrate numbers of admissible strategies. The first system is single plurality. The second system is $s=\{1, m-1\}$, in which a voter can

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vote for either one or all but one candidate. Although the latter system may be unfamiliar, we note that it is equivalent to the negative voting system examined by Brams (1977), which seems first to have been proposed by Boehm (1976). Under negative voting each voter is allowed to cast a vote on one candidate. This vote can be either *for* or *against* the candidate. A *for* vote adds one point to the candidate's score, and an *against* vote subtracts one point from the candidate's score. The outcome of a negative voting system ballot is the subset of candidates with the largest *net vote* total (sum of for and against points), which may be negative. Since a vote against a candidate has the same ultimate effect as a vote for every other candidate, the negative voting system is tantamount to system $\{1, m-1\}$. When $m=3$, this system is equivalent to approval voting, but for $m>3$ negative voting has fewer feasible strategies than approval voting. In what follows we shall refer to system $\{1, m-1\}$ as the negative voting system.

The following corollaries of Theorem 2 identify the admissible strategies of a concerned voter under single plurality and under negative voting. Recall that $L(P)$ is the voter's subset of least-preferred candidates. In Corollary 4, $\bar{a}$ denotes the strategy in which the voter votes for all candidates other than candidate a (or casts a vote *against* candidate a).

COROLLARY 3. *Strategy* $\{a\}$ *is admissible for single plurality and concerned* P *if and only if* a *is not in* L(P).

COROLLARY 4. *Suppose* $m \geq 3$. *Then strategy* $\{a\}$ *is admissible for negative voting and concerned* P *if and only if the voter strictly prefers* a *to at least two other candidates, and strategy* $\bar{a}$ *is admissible for negative voting and concerned* P *if and only if the voter strictly prefers at least two other candidates to* a.

Corollaries 1, 3 and 4, which provide necessary and sufficient conditions for admissible strategies under three different simple voting systems, can be used to identify and compare sets of admissible strategies for various preference relations on the candidates. For example, given the trichotomous preference order $(ab)cd$ on the set $\{a, b, c, d\}$ of four candidates, the sets of admissible strategies for approval voting, single plurality, and negative voting are respectively $\{\{a, b\}, \{a, b, c\}\}$, $\{\{a\}, \{b\}, \{c\}\}$ and $\{\{a\}, \{b\}, \bar{c}, \bar{d}\}$.

The numbers of admissible strategies for all concerned P orders on four candidates are shown in Table 1. It is clear that the relative numbers of admissible strategies for the three systems are very sensitive to the specific form of P. Although approval voting may offer more admissible strategies than other systems, as when P is $a(bc)d$, it may also offer fewer admissible strategies than the others. Hence it is not generally true that, by comparison to other voting systems, approval voting will overwhelm the voter with a wealth of *viable* options.

Table 1. Numbers of Admissible Voting Strategies for Three Voting Systems with Four Candidates

Concerned Preference Order		Number of Admissible Strategies for: Approval Voting	Negative Voting	Single Plurality
dichotomous	$a(bcd)$	1	1	1
	$(abc)d$	1	1	3
	$(ab)(cd)$	1	4	2
trichotomous	$(ab)cd$	2	4	3
	$ab(cd)$	2	4	2
	$a(bc)d$	4	2	3
multichotomous:	$abcd$	4	4	3

3. Sincere Voting and Strategyproofness

We now present a general comparison of approval voting and all other simple voting systems that is based on the following notions of sincere strategies and strategyproofness.

DEFINITION 5. Let P be a concerned preference order on the candidates. Then strategy S is *sincere* for P if and only if S is high for P; voting system s is *sincere* for P if and only if all admissible strategies for s and P are sincere; and voting system s is *strategyproof* for P if and only if exactly one strategy is admissible for s and P (in which case this strategy must be sincere).

Sincere strategies are essentially strategies that directly reflect the true preferences of a voter, i.e., that do not report preferences "falsely." For example, if P is $abcd$, then $\{a, c\}$ is not sincere since a and c are not the voter's two most-preferred candidates. Since it is generally felt that a democratic voting system should base the winner of an election on the true preferences of the voters, sincere strategies are of obvious importance to such systems.

They are also important to individual voters because, if a system is sincere, voters will always vote for *all* candidates ranked above the lowest-ranked candidates included in their chosen admissible strategies. (To illustrate when this proposition is not true, if P is $abcd$, $\{a, c\}$ is admissible under approval voting but obviously not sincere since this strategy involves voting for candidate c without also voting for preferred candidate b.) Thus, if

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a candidate voted for by a sincere voter should win, the sincere voter can rest assured that s/he could not have brought about the election of a preferred candidate by choosing a different admissible strategy.

A voting system that encourages sincere voting, it seems, would probably produce higher voter turnout. By allowing voters to tune their preferences more finely, and by forcing them less often to make insincere choices for strategic reasons, approval voting may well stimulate more voters to express themselves at the polls and enhance their attitudes towards the system.

Using Corollaries 1, 3 and 4, we can easily verify that, for the seven prototype preference orders on four candidates given in Table 1, approval voting is sincere in six cases (only *abcd* is excluded), negative voting is sincere in four cases, and single plurality is sincere in only the first three cases. In fact, it is no accident that approval voting is "more sincere" than both of the other systems used for Table 1: as the following theorem demonstrates, approval voting is the uniquely most sincere system of all the simple voting systems identified in Definition 3.

THEOREM 3. *If* P *is dichotomous then every voting system* s *is sincere for* P. *If* P *is trichotomous then the approval voting system is sincere for* P, *and this is the only system that is sincere for every trichotomous* P. *If* P *is multichotomous then no voting system* s *is sincere for* P.

No system is sincere when P is multichotomous because, for every s and every P with four or more indifference subsets, there is an admissible strategy that is not sincere. When there are relatively few candidates in a race, however, it is reasonable to expect many voters to have dichotomous or trichotomous preference orders. Theorem 3 tells us that when voters do not (or cannot) make finer distinctions, approval voting is the most sincere of all single-ballot nonranked voting systems. Its pertinence to elections today is evident.

Even if a voting system is sincere for P, however, it is not strategyproof for P if it allows more than one admissible strategy. Although manipulation of election outcomes by strategic voting in multicandidate elections is an old topic, only recently has it been shown, through the work of Gibbard (1973, 1977), Satterthwaite (1975), Gärdenfors (1976), Kelly (1977), Blin and Satterthwaite (1977), Barberá (1977a, 1977b), and Pazner (1978), that virtually every type of reasonable election method is subject to the influence of strategic voting; very recently, however, Kalai and Muller (1977), Peleg (1978), and Dutta and Pattanaik (1978) have offered new arguments for the robustness of many voting systems.

Like sincerity, strategyproofness seems a desirable property for a voting system to have. If voters have only one admissible strategy, they will never have an incentive to deviate from it even if they know the result of voting by all the other voters.

Since the demands of strategyproofness are more stringent than those for sincere voting, the circumstances that imply strategyproofness are less likely to obtain than the circumstances that imply sincerity. Nevertheless, as with sincerity, the approval voting system is the uniquely most strategyproof of all systems covered by Definition 3.

THEOREM 4. *If* P *is dichotomous then the approval voting system is strategyproof for* P, *and this is the only system that is strategyproof for every dichotomous* P. *If* P *is trichotomous or multichotomous then no voting system* s *is strategyproof for* P.

The second part of Theorem 4 simply says that if P has at least three indifference subsets, then any system s has at least two strategies that are admissible for s and P; hence, s can be manipulated by a choice of one and not another admissible strategy. The first part of Theorem 4 depends on Corollary 2: if P is dichotomous, then $M(P)$ is the unique admissible strategy for approval voting. In addition, if s is not the approval voting system, then there is a dichotomous P such that there are at least two admissible strategies for s and P.

Theorems 3 and 4 provide very strong support for approval voting evaluated against the important criteria of sincerity and strategyproofness, which are investigated further by Fishburn (1978), and, in a cardinal-utility context, by Merrill (1978). (Other properties that approval voting satisfies are delineated in different axiomatizations of approval voting in Fishburn [1977a, 1977c].) In section 4 we consider the propensity of approval voting and other voting systems to elect a candidate preferred by a majority of voters to every other candidate.

4. Dichotomous Preferences

We now demonstrate that, within the context of single-ballot nonranked voting, approval voting is the uniquely best system with respect to Condorcet's criterion of majority rule when preferences are dichotomous. Many of the results in this section have been noted previously by Brams (1976, 1978a, 1978c).

Throughout this section we shall let V denote a finite list of preference orders on the candidates, with each term in the list representing the preference of a particular voter. Clearly, each allowable preference order may appear more than once in V, or not at all. The *Condorcet candidates* with

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respect to any given V are the candidates in the set Con(V), where

Con(V) = {a: for each candidate $b \neq a$, at least as many terms in V have a preferred to b as have b preferred to a}.

Thus candidate a is in Con(V) if and only if as many voters prefer a to b as prefer b to a for each b other than a.

As is well known, Con(V) can be empty, which occurs for example if $V = (abc, cab, bca)$, i.e., there is a voters' paradox or cyclical majorities. Condorcet's basic majority rule asserts that candidate a wins the election when Con(V) contains only a and no other candidate. Since it is possible for more than one candidate to be in Con(V), as when the same number of voters prefer a to b as prefer b to a, we extend Condorcet's rule to assert that if Con(V) is not empty, then some candidate in Con(V) wins the election. In particular, this will be the case for a single-ballot voting system if every candidate who is in the outcome from the ballot is in Con(V), provided it is not empty.

It has been shown by Inada (1964) that if all preference orders in V are dichotomous, then Con(V) is not empty. Using the results of the preceding sections, which presume Assumptions P and R, we shall prove much more than this, namely that the use of admissible strategies in approval voting when preferences are dichotomous always yields Con(V) as the outcome. Moreover, we shall show that, for any other single-ballot nonranked voting system s, the use of admissible strategies under dichotomous preferences can give an outcome that contains no candidate in Con(V).

The following formulation will be used to express these results more rigorously.

DEFINITION 6. For any finite list V of preference orders on the candidates, and for any voting system s as identified in Definition 3, let $V(s)$ be the set of all functions that assign an admissible strategy to each of the terms in V. For each function α in $V(s)$, let $F(\alpha)$ be the outcome (the set of candidates with the greatest vote total) when every voter uses the admissible strategy that is assigned to his preference order by α.

As an illustration, assume $V = (abc, abc, c(ab))$, consisting of two voters who prefer a to b to c and one voter who is indifferent between a and b but prefers c to both a and b. If s is the single plurality system, then $V(s)$ contains $(2)(2)(1) = 4$ functions since each of the first two voters has two admissible strategies and the third voter has one admissible strategy according to Corollary 3. The outcomes for the four α functions are $\{a\}$, $\{a, b, c\}$, $\{a, b, c\}$ and $\{b\}$. $F(\alpha) = \{a, b, c\}$, for example, if α assigns strategy $\{a\}$ to the first voter and strategy $\{b\}$ to the second voter.

THEOREM 5. *Suppose all preference orders in* V *are dichotomous and* s *is the approval voting system. Then* $F(\alpha) = \mathrm{Con}(V)$ *for every* $\alpha \in V(s)$.

Hence, under approval voting and dichotomous preferences, the use of admissible strategies invariably yields Con(V) as the outcome.

To illustrate the dichotomous preferences situation for single plurality, suppose that the candidate set is $\{a, b, c\}$ and there are $2N+1$ terms in V, one of which is $a(bc)$ with the other $2N$ terms divided evenly between $b(ac)$ and $(ac)b$. Then Con(V) = $\{a\}$. However, if as many as two of the N voters who have the order $(ac)b$ vote for c, then $F(\alpha) = \{b\}$ and $F(\alpha)$ and Con(V) are therefore disjoint. The following theorem shows that a similar result holds for every system other than the approval voting system.

THEOREM 6. *Suppose* s *is a voting system as described in Definition 3 and* s *is not the approval voting system. Then there exists a* V *consisting entirely of dichotomous preference orders and an* $\alpha \in V(s)$ *such that no candidate in* $F(\alpha)$ *is in* Con(V).

Hence, among all single-ballot nonranked voting systems, there is a uniquely best system by Condorcet's criterion when preferences are dichotomous. Since approval voting is both strategyproof and selects Con(V) when voters use admissible strategies and have dichotomous preferences, it is the best possible voting system in two important respects when preferences are dichotomous.

In contrast to the definitive picture obtained for dichotomous preferences, comparisons among approval voting and other single-ballot systems are much less clear when some voters divide the candidates into more than two indifference subsets. The main work to date on the propensities of different single-ballot systems to elect a Condorcet candidate when Con(V) is not empty has been reported in Fishburn (1974) and Fishburn and Gehrlein (1976; 1977). The conclusions in these studies are based primarily on computer simulations to estimate the probabilities that various voting systems will elect the Condorcet candidate when there is a unique such candidate. Without going into details, we may draw the general conclusion from these studies that when all voters have linear preference orders and vote sincerely, the propensity of the approval voting system to elect the Condorcet candidate is comparable to if not better than the propensities of other single-ballot nonranked systems to elect the Condorcet candidate. In conjunction with the dichotomous preferences results of the preceding section, this strongly suggests that approval voting compares favorably with other single-ballot systems on the basis of Condorcet's rule. Additional theoretical

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support for this view is provided by Weber (1977a; 1977b) using a probabilistic model.[1]

5. Approval Voting and Presidential Elections

Several desirable properties of approval voting is multicandidate elections have been described in previous sections. As a practicable reform, Kellett and Mott (1977) have made a strong case that approval voting be adopted in presidential primaries, which, at least in the early stages, often involve several candidates running for their party's nomination. When Kellett and Mott asked a sample of 225 Pennsylvania voters to "vote for any candidates whose nomination you can support" in the 1976 presidential primary (eight Democratic candidates and eight Republican candidates were listed on two sample ballots), 72 percent of those voting chose to support two, three, or four candidates.

A case for approval voting in national party conventions can also be made. As in primaries, the main effect would probably be to give comparatively more support to moderates that most delegates find acceptable, comparatively less to extremists who are only acceptable to ideological factions in their party.

If there had been approval voting in the 1972 Democratic convention, it seems at least doubtful that George McGovern would have been his party's nominee. Not only did he not have strong support from his party rank and file (Keech and Matthews, 1976, p. 212), but also he was not accorded any reasonable chance of winning in the general election.

Although most general elections are, for all intents and purposes, two-candidate contests, since 1900 there have been several serious bids by third-party candidates in presidential elections (Mazmanian, 1974).[2] The most notable challenges in the first quarter of the century were in 1912, when Theodore Roosevelt won 27.4 percent of the popular vote, and in 1924, when Robert La Follette won 16.6 percent of the popular vote.

More recently, Harry Truman faced defections from both wings of the Democratic Party in 1948. The Progressive party candidate, Henry Wallace, and the States' Rights party candidate, Strom Thurmond, each captured 2.4 percent of the popular vote. Nevertheless, Truman was able to win 49.6 percent of the popular vote to Republican Thomas Dewey's 45.1 percent.

The most serious challenge by a third-party candidate since World War II was that of George Wallace in the 1968 presidential election. We shall shortly analyze this election in some detail to try to assess the possible effects of approval voting in a presidential election.

Most recently, Eugene McCarthy ran as a third-party candidate in the 1976 presidential election. Playing the spoiler role, McCarthy sought to protest what he saw to be the outmoded procedures and policies of the Democratic party, for whose nomination he had run in 1968 and 1972. Although McCarthy garnered only 0.9 percent of the popular vote, his candidacy may have cost Jimmy Carter four states, which Gerald Ford won by less than what McCarthy polled. In the end, of course, Carter did not need the electoral votes of these states, but had he lost in a few states that he won by slim margins, these McCarthy votes could have made the difference.

We turn now to an analysis of the third-party challenge by George Wallace's American Independence party in 1968. As with Strom Thurmond's support 20 years earlier, Wallace's support was concentrated in the South. Although Wallace had no reasonable chance of winning the presidency, it seems at the time that he had a very good chance of preventing both Richard Nixon and Hubert Humphrey from winning a majority of electoral votes, thereby throwing the election into the House of Representatives. There Wallace could have bargained with these candidates for major policy concessions—in particular, weaker enforcement of civil rights statutes and a halt to busing.

Wallace captured 13.5 percent of the popular vote and was the victor in five states, winning 46 electoral votes. He came close to denying Nixon, who got 43.4 percent of the popular vote to Humphrey's 42.7 percent, an electoral-vote majority.

[1] Recently we (Brams and Fishburn, 1978) have been able to make these arguments more conclusive by proving that (i) among all single-ballot nonranked voting systems, approval voting is the only system that guarantees the existence of sincere admissible strategies that elect a Condorcet candidate (if one exists); (ii) allowing for a runoff election between the top two candidates under plurality voting also ensures the existence of admissible strategies that elect a Condorcet candidate, but they are not necessarily sincere; (iii) if plurality voting (with or without a runoff) leads to the election of a Condorcet candidate whatever admissible strategies voters choose, so does approval voting, but the reverse is not true: there are situations in which approval voting guarantees the election of a Condorcet candidate but plurality voting (with or without a runoff) does not. The last statement says, in effect, that approval voting does a better job of *ensuring* the election of a Condorcet candidate than its main competitors, single-ballot and two-ballot plurality voting; we consider this result on a par with our sincerity and strategyproofness results in that it establishes the "dominance" of approval voting—but in this case with respect to the Condorcet criterion.

[2] An interesting model to explain why "sophisticated" or "disillusioned" voters support third-party candidates under single plurality systems is given in Riker (1976); an alternative model positing "sincere" voters is given in Brams (1978b).

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Would this outcome have been different, or would its magnitude have significantly changed, if there had been approval voting in 1968? Presumably, all voters who voted for one of the three candidates would not have changed their votes. But how many would have cast second approval votes, and for whom would they have voted?

The best information available to answer this question was collected in the University of Michigan Survey Research Center's 1968 National Election Study. Data derived from a "feeling thermometer" assessment of candidates—whereby respondents are asked to indicate warm or cold feelings toward the candidates on a 100-degree scale—may be used to define an "acceptability" scale for candidates, from which plausible approval voting strategies of voters can be surmised.

Taking account of both the reported votes of the respondents (the survey was taken just after the election) and their feeling-thermometer assessments for the candidates, Kiewiet (1977) developed a set of rules for assigning approval votes to respondents.[3] After adjusting reported voting by the sample to reflect the actual voting results, Kiewiet estimated that Nixon would have increased his vote total to 69.8 percent (a 58 percent increase over the 44.1 percent in the survey who reported voting for Nixon), Humphrey would have increased his vote total to 60.8 percent (a 44 percent increase over the 42.3 percent in the survey who reported voting for Humphrey), and Wallace would have increased his vote total to 21.3 percent (a 58 percent increase over the 13.5 percent in the survey who reported voting for Wallace.)

Kiewiet draws several conclusions from his analysis. First, single plurality voting nearly deprived Nixon of his victory: although many voters were certainly not wildly enthusiastic about Nixon, more than a two-thirds majority probably considered him at least acceptable. Second, although most of the additional approval votes Nixon and Humphrey would have received would have come from each other's supporters, Wallace supporters—according to the rules used for assigning approval votes—would have cast more than twice as many approval votes for Nixon as for Humphrey.

It is this factor which largely explains Nixon's 9 percent approval-voting edge over Humphrey. Wallace also would have benefited from approval voting. In fact, his estimated 21-percent approval voting share exactly matches the percentage who reported they would vote for him two months before the election (Scammon and Wattenberg, 1970, pp. 170–71). If there had been approval voting, Wallace almost surely would not have lost most of his original supporters, and probably would have picked up some support from the major-party voters as well, to capture approval votes from more than one-fifth of the electorate.

Perhaps the most interesting conclusion we can derive from these estimates is that Nixon was undoubtedly the Condorcet winner. Kiewiet estimates that he would have defeated Wallace in a pairwise contest 81.5 percent to 18.5 percent and would have defeated Humphrey 53.4 percent to 46.6 percent, given the propensity of Wallace voters to favor Nixon.

Several objections can be raised against Kiewiet's estimates and indeed against virtually any estimates based on assumptions about how the attitudes or "feelings" of voters would translate into voting behavior. Rather than dwelling on these, however, let us consider a rather different set of estimates made by Kiewiet based on more "strategic" assumptions.

These assumptions reflect the view of most voters in 1968 that only Humphrey and Nixon stood a serious chance of winning the election. After all, even at his high point in the polls, Wallace commanded the support of barely more than one-fifth of the electorate. It is plausible to assume, therefore, that voters would cast approval votes to distinguish between Humphrey and Nixon (Brams, 1978a, 1978b).

More specifically, Kiewiet assumed that (i) Humphrey and Nixon supporters would vote for Wallace, if they also approved of him, but would not vote for the other major-party candidate; (ii) all Wallace voters would vote for either Humphrey or Nixon, but not both, in addition to Wallace. As he put it,

> In effect, a poll indicating Wallace had no chance of winning would, under approval voting, turn the election into two elections: the first, a pairwise contest between Nixon and Humphrey, wherein all voters would choose one or the other; the second, a sort of referendum for Wallace, who would receive approval votes from voters who wished to support him even if he could not win the election.

In operational terms, Kiewiet postulated that Humphrey and Nixon supporters would vote for their first choice and, in addition, for Wallace if the latter's thermometer rating exceeded 50. Wallace supporters, on the other hand, were assumed always to cast a second approval vote for the major-party candidate they gave the highest thermometer rating to, no matter what this rating was. Thereby Wallace voters were "forced" to be rational in accordance with the assumptions of the poll model.

What estimates does this set of assumptions yield? Nixon would have received 53.4 percent of the popular vote and Humphrey 46.6 percent—the same percentages given earlier had they been

[3] For a related effort, applied to primaries, to translate 1972 feeling-thermometer data into electoral outcomes under a variety of decision rules, see Joslyn (1976).

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in a pairwise contest—and Wallace 21.3 percent. Thus, the approval voting percentages of Humphrey and Nixon would have been substantially reduced over those estimated earlier (69.8 and 60.8 percent, respectively), but Wallace would have come out exactly the same (21.3 percent estimated earlier) since the "strategic" assumptions do not alter the voting behavior of Wallace supporters for Wallace.

The two sets of estimates for Humphrey and Nixon probably bracket the percentages the candidates would actually have received had there been approval voting in 1968. Whichever set gives the better estimate, Nixon in either case would have been the clear-cut winner in the popular-vote contest because of the much broader support he, rather than Humphrey, would have received from Wallace supporters.

The Electoral College also magnified Nixon's narrow popular-vote victory because he won by slim margins in several large states. However, speaking normatively, we believe this fact should have no bearing on the outcome. Much more significant is the fact that Nixon was the first or second choice of most voters and hence more acceptable than any other candidate. This, we believe, is the proper criterion for the selection of a president—and other democratically elected officials as well.

It is also interesting to note that approval voting would probably obviate the need for a runoff election in most multicandidate presidential elections if the Electoral College were abolished. No winning candidate in a presidential election has ever received less than 40 percent of the popular vote, with the exception of Abraham Lincoln in 1860, who got 39.8 percent. It seems highly unlikely that a candidate who is the first choice of 40 percent of the electorate would not be approved of by as many as one-sixth of the remaining voters and thereby receive at least 50 percent support from the electorate.

The legitimacy of election outcomes in the eyes of voters would certainly be enhanced if the winning candidate received the support of a majority of the electorate. This would be true even if the winner were the first choice of fewer voters than some other candidate, because this fact would not show up in the approval-voting returns.

By comparison, the proposed popular-vote amendment to abolish the Electoral College provides for a runoff between the top two vote-getters if neither receives at least 40 percent of the vote. This seems an unnecessary provision if more than 50 percent approve of the winning candidate. Of course, if no candidate wins even a majority of approval votes, then a runoff can still be conducted to ensure a majority winner.

But this would probably not be necessary in most presidential elections unless approval voting itself produces major changes in candidate strategies and election outcomes. Beyond these changes, however, approval voting could effect a fundamental alteration in the two-party system itself by encouraging additional parties or candidates to enter the fray. Fringe candidates, it seems, would probably drain little support from centrist candidates because, for strategic reasons, fringe candidate supporters would probably also tend to vote for a centrist. Additional centrist candidates, on the other hand, might draw support away from major-party candidates if they (the new centrists) were perceived as serious contenders.

The question that is hard to answer, in the absence of experience, is whether such contenders could position themselves in such a way as to displace the major-party candidates. If so, presumably they would be motivated to run, giving voters more viable alternatives from which to choose and, in the process, weakening the two-party system. Their election, however, would probably not produce drastic changes in public policy since they would not be viable if they were unacceptable to numerous middle-of-the-road voters.

Barring unforeseen changes, it seems likely that at the same time approval voting would give some additional support to strong minority candidates like George Wallace, it would also help centrist candidates—including perhaps nominees of new parties—both in winning their party's nomination in the primaries and conventions and prevailing against more extreme candidates in the general election. Coupled with the greater opportunity it affords voters to express their preferences, and the greater likelihood it provides the winning candidate of obtaining majority support, approval voting would seem to be an overlooked reform that now deserves to be taken seriously.

In a way, approval voting is a compromise between plurality voting and more complicated schemes like the Borda count which require voters to rank candidates. In our view, the latter schemes are both too complicated and unnecessary in elections in which there is only a single winner. (Elections in which there are multiple winners, such as to a committee or council, would also seem well suited for approval voting, but that is a subject for another article). On the other hand, approval voting is not only quite easy to understand—even if some of its theoretical implications are not so obvious—but it also seems an eminently practicable scheme that could readily be implemented on existing voting machines.

Appendix

Proofs of the technical results of the paper are given in this appendix. It will be assumed through-

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out that P is concerned and that Assumptions P and R hold. The following notation will be used in addition to the notation introduced in the paper. The empty set is denoted by $\varnothing$; $A \cap B = \{a\colon a \in A$ and $a \in B\}$, the intersection of A and B; $A \subseteq B$ means that A is a subset of B; $A \nsubseteq B$ means that A is not a subset of B (something in A is not in B); $A \subset B$ means that A is a proper subset of B($A \subseteq B$ and something in B is not in A); and $|A|$ is the number of candidates in A.

THEOREM 1. S *dom* T *for* P *if and only if* S $\neq$ T, S\T *is high for* P, T\S *is low for* P, *and neither* S\T *nor* T\S *is the set of all candidates.*

Proof. Suppose first that $S \neq T$ and that the other conditions on S and T stated thereafter in the theorem are true. Then there must be a and b such that aPb and either (i) $a \in S\backslash T$ and $b \notin S\backslash T$, or (ii) $a \notin T\backslash S$ and $b \in T\backslash S$. For example, if $S\backslash T \neq \varnothing$ let a be a most preferred candidate in $S\backslash T$ so that $a \in M(P)$ since $S\backslash T$ is high. If it were then true that aPb never held for $b \notin S\backslash T$, then aPb implies $b \in S\backslash T$ and, since $S\backslash T$ is high, this would require $S\backslash T$ to be the entire set of candidates, which contradicts one of our hypotheses. Therefore, when $S\backslash T \neq \varnothing$, aPb for some $a \in S\backslash T$ and $b \notin S\backslash T$. Similarly, if $T\backslash S \neq \varnothing$ then aPb for some $b \in T\backslash S$ and $a \notin T\backslash S$. Since $S \neq T$ implies either $S\backslash T \neq \varnothing$ or $T\backslash S \neq \varnothing$, either (i) or (ii) must be true.

Let f be a contingency for which $f(a) = f(b)$ with $f(a) > f(c) + 1$ for all other candidates c when (i) or (ii) holds. Then

$F(S, f) = \{a\}$ and $F(T, f) = \{b\}$ if $a \in S\backslash T$ and $b \in T\backslash S$;
$F(S, f) = \{a, b\}$ and $F(T, f) = \{b\}$ if either $a \in S\backslash T$ and $b \in S \cap T$ or $b \in T\backslash S$ and $a \notin S \cup T$;
$F(S, f) = \{a\}$ and $F(T, f) = \{a, b\}$ if either $a \in S\backslash T$ and $b \notin S \cup T$ or $b \in T\backslash S$ and $a \in S \cap T$.

Since the enumerated possibilities for a and b cover all possibilities under (i) and (ii), Assumption P and aPb imply $F(S, f)PF(T, f)$. Hence there is a contingency under which the outcome for S is strictly preferred to the outcome for T.

Continuing with the assumptions that $S \neq T$, $S\backslash T$ is high for P, and $T\backslash S$ is low for P, we show next that $F(S, f)RF(T, f)$ for all contingencies f. Under the given assumptions,

$$a \notin T\backslash S \text{ and } b \in T\backslash S \text{ imply } aRb, \tag{1}$$

$$a \in S\backslash T \text{ and } b \notin S\backslash T \text{ imply } aRb. \tag{2}$$

We consider three exhaustive possibilities for a contingency f.

Case 1: $F(T, f) \cap (S\backslash T) \neq \varnothing$. It then follows that $F(S, f) = F(T, f) \cap (S\backslash T)$. To apply Assumption R, let $A = \varnothing$, $B = F(S, f)$ and $C = F(T, f)\backslash F(S, f)$. If $C = \varnothing$, then $F(S, f) = F(T, f) = B$, and BRB by Assumption R. If $C \neq \varnothing$, then each $b \in B$ is in $S\backslash T$ and each $c \in C$ is not in $S\backslash T$ so that bRc by (2); therefore $BR(B \cup C)$, or $F(S, f)RF(T, f)$, by Assumption R.

Case 2: $F(S, f) \cap (T\backslash S) \neq \varnothing$. Then $F(T, f) = F(S, f) \cap (T\backslash S)$. In this case, let $A = F(S, f)\backslash F(T, f)$, $B = F(T, f)$ and $C = \varnothing$ for direct application of Assumption R. If $A = \varnothing$, then $F(S, f) = F(T, f) = B$ with BRB. And if $A \neq \varnothing$, then each $a \in A$ is not in $T\backslash S$ and each $b \in B$ is in $T\backslash S$ so that aRb by (1); therefore $(A \cup B)RB$, or $F(S, f)RF(T, f)$, by Assumption R.

Case 3: Both $F(T, f) \cap (S\backslash T)$ and $F(S, f) \cap (T\backslash S)$ are empty. Hence if $a \in F(T, f)$ then $a \notin S\backslash T$, and if $a \in F(S, f)$ then $a \notin T\backslash S$. Suppose $a \in F(T, f)$ and $a \notin T\backslash S$. Then a is in both T and S or else in neither T nor S, and in each case it follows that $a \in F(S, f)$. Similarly, if $a \in F(S, f)$ and $a \notin S\backslash T$, then $a \in F(T, f)$. Therefore, the set of all candidates in $F(T, f)$ but in neither $S\backslash T$ nor $T\backslash S$ is identical to the set of all candidates in $F(S, f)$ but in neither $S\backslash T$ nor $T\backslash S$. Let B be this common set, and let $A = F(S, f) \cap (S\backslash T)$ and $C = F(T, f) \cap (T\backslash S)$ so that $F(S, f) = A \cup B$ and $F(T, f) = B \cup C$. Since (1) and (2) imply that aRb, bRc and aRc for all $a \in A$, $b \in B$ and $c \in C$, it follows from Assumption R that $F(S, f)RF(T, f)$.

Thus far in this proof we have shown that if the latter conditions on S and T in Theorem 1 are true then S dom T. We now establish the necessity of these conditions for dominance. If either $S\backslash T$ or $T\backslash S$ is the set of all candidates, then one of S and T must be the set of all candidates and the other must be empty, in which case $F(S, f) = F(T, f)$ for all f so that neither strategy can dominate the other. A similar conclusion holds if $S = T$. If either $S\backslash T$ is not high for P or $T\backslash S$ is not low for P then there are a and b such that aPb and either (iii) $a \in T\backslash S$ and $b \notin T\backslash S$ or (iv) $a \notin S\backslash T$ and $b \in S\backslash T$. Then, by the first two paragraphs of this proof (interchange S and T), there is an f such that $F(T, f)PF(S, f)$, and therefore S cannot dominate T.

THEOREM 2. *Strategy* S *is admissible for* s *and* P *if and only if* S *is feasible for* s *and either:*

C1: M(P) $\subseteq$ S *and* S *does not include a nonempty proper subset* B *that is low for* P *and has* S\B *feasible for* s; *or*

C2: L(P) $\cap$ S $= \varnothing$ *and there is no nonempty* A *that is high for* P, *has* A $\cap$ S $= \varnothing$ *and for which* A $\cup$ S *is feasible for* s.

Proof. Given that S is feasible for s we are to show that it is admissible for s and P if and only if $C1$ or $C2$ holds. Suppose first that $M(P) \nsubseteq S$ and $L(P) \cap S \neq \varnothing$, and take $a \in S\backslash M(P)$ and $b \in L(P) \cap S$. Form T from S by deleting b and adding a so that $|T| = |S|$ with T feasible. Then $T\backslash S = \{a\}$, which is

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high, and $S \backslash T = \{b\}$, which is low. Hence, by Theorem 1, T dom S and S is not admissible. Therefore, if S is admissible, it must be true that either $M(P) \subseteq S$ or $L(P) \cap S = \emptyset$.

Suppose next that $M(P) \subseteq S$. If S includes a nonempty proper subset B that is low for P and has $S \backslash B$ feasible for s, then, by Theorem 1, $(S \backslash B)$ dom S since $(S \backslash B) \backslash S = \emptyset$, which is high, and $S \backslash (S \backslash B) = B$, which is low, and therefore S is not admissible. On the other hand, suppose T is feasible and T dom S. Then $T \neq S$ and $T \backslash S$ high require $T \backslash S = \emptyset$ since $M(P) \subseteq S$. Therefore $T \subset S$. If $T = \emptyset$ then $S \backslash T$, which equals S, cannot be low since S cannot be the entire set of candidates (by Theorem 1) and since $M(P) \subseteq S$. Therefore T dom S requires $\emptyset \subset T \subset S$. Let B in the statement of $C1$ in Theorem 2 equal $S \backslash T$. Then B is a nonempty proper subset of S, B is low for P (by Theorem 1 and T dom S), and $S \backslash B = T$ is feasible for s. Therefore, given S feasible and $M(P) \subseteq S$, S is not admissible if, and only if, the latter conditions in $C1$ are false.

Finally, suppose that $L(P) \cap S = \emptyset$. If there is a nonempty A that is high, disjoint from S, and for which $A \cup S$ is feasible for s, then $(A \cup S)$ dom S by Theorem 1 and therefore S is not admissible. (The possibility of $S = \emptyset$ and A being the set of all candidates is ruled out by the exclusion of m from s.) Conversely, suppose T is feasible and T dom S. Then, since $S \backslash T$ is low, $T \neq S$, and $L(P) \cap S = \emptyset$, we must have $S \backslash T = \emptyset$ and therefore $S \subset T$. Let A in the statement of C2 equal $T \backslash S$. Then A is nonempty, A is high for P (since $T \backslash S$ is high by Theorem 1), $A \cap S = \emptyset$, and $A \cup S = T$ is feasible for s. Therefore, given S feasible and $L(P) \cap S = \emptyset$, S is not admissible if, and only if, the latter conditions in $C2$ are false.

COROLLARY 1. *Strategy* S *is admissible for approval voting and* P *if, and only if,* $\mathrm{M(P)} \subseteq \mathrm{S}$ *and* $\mathrm{L(P)} \cap \mathrm{S} = \emptyset$.

Proof. Suppose first that $M(P) \nsubseteq S$. Take $a \in S \backslash M(P)$. Then $(S \cup \{a\})$ dom S by Theorem 1 so that S is not admissible (for approval voting and P). Suppose next that $L(P) \cap S \neq \emptyset$. Take $b \in L(P) \cap S$. Then $(S \backslash \{b\})$ dom S by Theorem 1 so that S is not admissible. Admissibility therefore requires both $M(P) \subseteq S$ and $L(P) \cap S = \emptyset$. Assume then that $M(P) \subseteq S$ and $L(P) \cap S = \emptyset$. Since $L(P) \cap S = \emptyset$ implies that there is no nonempty proper subset B of S that is low for P, and $M(P) \subseteq S$ implies that there is no nonempty A that is high for P and disjoint from S, it follows from Theorem 2 that S is admissible.

COROLLARY 2. *There is a unique admissible strategy for approval voting and* P *if and only if* P *is dichotomous, and this unique strategy is* M(P).

Proof. The proof follows immediately from Corollary 1.

COROLLARY 3. *Strategy* $\{\mathrm{a}\}$ *is admissible for* $s = \{1\}$ *and* P *if and only if* $\mathrm{a} \notin \mathrm{L(P)}$.

Proof. By Theorem 2, the only way $\{a\}$ cannot be admissible for single plurality and P is when $a \in L(P)$. Then both $C1$ and $C2$ fail. If $a \notin L(P)$ then $C2$ is true.

COROLLARY 4. *Suppose* $\mathrm{m} \geq 3$. *Then* $\{a\}$ *is admissible for* $\{1, \mathrm{m}-1\}$ *and* P *if and only if* a *is strictly preferred to at least two other candidates, and* $\bar{\mathrm{a}}$ *is admissible for* $\{1, \mathrm{m}-1\}$ *and* P *if and only if at least two other candidates are strictly preferred to* a.

Proof. Let $s = \{1, m-1\}$ with $m \geq 3$. First, if a is strictly preferred to two or more candidates then $L(P) \cap \{a\} = \emptyset$ and if nonempty A is high for P and $a \notin A$ then $1 < |A \cup \{a\}| \leq m-2$ so that $A \cup \{a\}$ is not feasible for s. Hence, by $C2$ of Theorem 2, $\{a\}$ is admissible. Second, if at least two candidates are preferred to a then $M(P) \subseteq \bar{a}$ and if B is a nonempty proper subset of $\bar{a}$ that is low then B does not contain any candidate preferred to a so that $2 \leq |S \backslash B| < m-1$, in which case $S \backslash B$ is not feasible for s. Hence, by $C1$ of Theorem 2, $\bar{a}$ is admissible. Third, suppose that a is preferred to at most one other candidate. If a is preferred to no other candidate then $L(P) \cap \{a\} \neq \emptyset$ and $C1$ and $C2$ fail; if a is preferred to exactly one candidate then $M(P) \nsubseteq \{a\}$, so $C1$ fails, and, even though $L(P) \cap \{a\} = \emptyset$, $C2$ is seen to fail when A equals all candidates except a and the one in $L(P)$. Hence $\{a\}$ is not admissible in this case. Finally, suppose that at most one other candidate is preferred to a. If nothing else is preferred to a, then $M(P) \nsubseteq \bar{a}$ and $C1$ and $C2$ fail; if exactly one candidate is preferred to a, then $L(P) \cap \bar{a} \neq \emptyset$, so $C2$ fails and, although $M(P)$ may be included in $\bar{a}$, if in fact $M(P) \subseteq \bar{a}$ then with B all candidates other than a and the one in $M(P)$ we see that B is a nonempty proper subset of $\bar{a}$ that is low for P and for which $\bar{a} \backslash B = M(P)$ is feasible, thus implying that $C1$ fails. Hence $\bar{a}$ is not admissible in this final case.

THEOREM 3. *If* P *is dichotomous then every* s *is sincere for* P. *System* s *is sincere for every trichotomous* P *if and only if* s *is the approval voting system. If* P *is multichotomous then no* s *is sincere for* P.

Proof. If P is dichotomous then every S that has $M(P) \subseteq S$ or $L(P) \cap S = \emptyset$ is sincere. Hence, by Theorem 2, every s is sincere for P. If P is trichotomous and s is the approval voting system then it follows immediately from Corollary 1 that s is sincere for P.

We show next that the trichotomous result for systems other than approval voting holds for each $m \geq 3$. For a generic trichotomous P with in-

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difference subsets A_1, A_2 and A_3 as in Definition 1, let $m_i = |A_i|$ with $m_1 + m_2 + m_3 = m$. If s is not the approval voting system then either there is an $i \in s$ with $1 \le i < m-1$ and $i+1 \notin s$, or there is an $i \in s$ with $1 < i \le m-1$ and $i-1 \notin s$. Suppose first that $i \in s$, $1 \le i < m-1$ and $i+1 \notin s$, and let P be such that $m_1 = i$, $m_2 = m-i-1$ and $m_3 = 1$. Form strategy S with $i-1$ candidates from A_1 and 1 candidate from A_2. Then S is not sincere, but it is admissible by $C2$ of Theorem 2. Suppose next that $i \in s$, $1 < i \le m-1$ and $i-1 \notin s$, and let P be such that $m_1 = i-1$, $m_2 = m-i$ and $m_3 = 1$. Form S with $i-1$ candidates from A_1 and 1 candidate from A_3: i.e., $S = A_1 \cup A_3$. Then S is not sincere, but it is admissible by $C1$ of Theorem 2.

For the multichotomous case let P have $n \ge 4$ indifference subsets $A_1, A_2, \cdots, A_n$ as in Definition 1, with $m_i = |A_i|$ and $m_1 + m_2 + \cdots + m_n = m$. In addition, let $s = \{s_1, s_2, \cdots, s_r\}$ with $1 \le s_1 < s_2 < \cdots < s_r \le m-1$ and $r \ge 1$ be a generic voting system for m-candidate elections. We are to show that s is not sincere for P. If $s_1 > m_1$ let S with $|S| = s_1$ include A_1, exclude something in A_2, and contain something in A_3. Then S is admissible by $C1$ but it is not sincere. Assume henceforth that $s_1 \le m_1$. Now suppose that $m_1 < s_i < m - m_n$ for some $s_i \in s$. Then form S with $|S| = s_i$ so that S includes A_1, excludes A_n, contains something in A_3 and does not contain everything in A_2. Then S is not sincere, but it is admissible by $C2$. Therefore, to avoid insincerity for s we require

$$s_i \le m_1 \quad \text{or} \quad m - m_n \le s_i \quad \text{for every } s_i \in s. \qquad (3)$$

Now for any $s_i \le m_1$ form S with $|S| = s_i$ with one candidate from A_2 and the rest (if any) from A_1. This S is insincere. To avoid admissibility for S by $C2$, it must be true that $s_i < s_j \le m_1 + m_2$ for some $s_j \in s$. However, (3) also requires $s_j \le m_1$ to avoid insincerity for s. Therefore, it follows that if s is to be sincere for P, then for every $s_i \le m_1$ there is an $s_j \le m_1$ such that $s_i < s_j$. Since $s_1 \le m_1$, this is impossible unless m_1 is infinite. But m_1 is finite. Therefore s is not sincere for P.

THEOREM 4. *System* s *is strategyproof for every dichotomous* P *if and only if* s *is the approval voting system. If* P *has more than two indifference subsets then no* s *is strategyproof for* P.

Proof. When P is dichotomous, with indifference subsets $M(P)$ and $L(P)$, it follows from Corollary 2 that approval voting is strategyproof for P. Suppose then that s is not the approval voting system. If $i \in s$, $1 \le i < m-1$ and $i+1 \notin s$, let P be dichotomous with $|M(P)| = i+1$. Then, by $C2$, every S that has $|S| = i$ and $S \subset M(P)$ is admissible for s and P, and therefore s is not strategyproof for P. On the other hand, if $i \in s$, $1 < i \le m-1$ and $i-1 \notin s$, let P be dichotomous with $|M(P)| = i-1$. Then, by $C1$, every S that has $|S| = i$ and $M(P) \subset S$ is admissible for s and P, and therefore s is not strategyproof for P.

Suppose next that P has indifference subsets $A_1, A_2, \cdots, A_n$ with $n \ge 3$, and let $m_i = |A_i|$ with $m_1 + m_2 + \cdots + m_n = m$. Let $s = \{s_1, \cdots, s_r\}$ with $1 \le s_1 < \cdots < s_r \le m-1$. If $s_1 > m_1$ then, by $C1$, every S with $|S| = s_1$ and $M(P) \subset S$ is admissible, and there is more than one such S. If $s_1 < m_1$ and $s_i \ne m_1$ for all $s_i \in s$, let s_j be the largest s_i that is less than m_1. Then, by $C2$, all S with $|S| = s_j$ and $S \subset A_1 = M(P)$ are admissible. Hence if $s_i \ne m_1$ for all $s_i \in s$ then there are at least two admissible strategies, and s is not strategyproof for P.

Continuing in the context of the preceding paragraph, if $s_r < m - m_n$ then, by $C2$, every S with $|S| = s_r$ and $S \cap A_n = \varnothing$ is admissible for s and P, and there is more than one such S. If $s_r > m - m_n$ and $s_i \ne m - m_n$ for all $s_i \in s$, let s_k be the smallest s_i that exceeds $m - m_n$. Then, by $C1$, every S with $|S| = s_k$ and $A_1 \cup A_2 \cup \cdots \cup A_{n-1} \subset S$ is admissible, and there is more than one such S. Hence if $s_i \ne m - m_n$ for all $s_i \in s$ then s is not strategyproof for P.

It then follows from the two preceding paragraphs that, if s is to be strategyproof for P, we must have m_1 and $m - m_n$ in s. However, if this is so, then $M(P) = A_1$ and $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ are both admissible for s and P according to Theorem 2. Hence no s can be strategyproof for P when $n \ge 3$.

THEOREM 5. *If all preference orders in* V *are dichotomous and* s *is the approval voting system, then* F(α) = Con(V) *for all* α∈V(s).

Proof. Given the theorem's hypotheses, Corollary 2 implies that V(s) consists of the unique function that assigns the subset of most-preferred candidates to each order in V. The outcome of this function must be Con(V) since as many terms in V have a preferred to b as have b preferred to a if, and only if, a gets as many votes as b.

THEOREM 6. *Suppose* s *is not the approval voting system. Then some* V *consisting entirely of dichotomous preference orders has an* α∈V(s) *such that* F(α)∩Con(V) = ∅.

Proof. Assume that s is not the approval voting system and let k_0 be an integer in $\{1, 2, \cdots, m-1\}$ that is not in s. Let V be a list of dichotmous preference orders such that for each P in this list $M(P)$ contains exactly k_0 candidates and $L(P)$ contains $m - k_0$ candidates. Then let $k^* \in s$ be such that there are admissible strategies for each P in V that contain exactly k^* candidates. Such a k^* must exist since it is not possible to have every strategy inadmissible for system s. If $k^* < k_0$ then any strategy for a voter that consists of k^* of the voter's most-preferred candidates will be admissible. And if $k^* > k_0$ then any strategy for a voter that contains the voter's k_0 most-preferred candidates and any $k^* - k_0$ of the voter's least-preferred candidates

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will be admissible. We now consider two cases according to whether $k^* < k_0$ or $k^* > k_0$.

Case 1: $k^* < k_0$. Construct V in the dichotomous format described above in such a way that candidate a is a most-preferred candidate for every voter and no other candidate is a most-preferred candidate for every voter. Then $\mathrm{Con}(V) = \{a\}$. Let $\alpha \in V(s)$ be such that a is never in the subset of k^* most-preferred candidates assigned to each voter by α. Then $F(\alpha)$ will not contain a.

Case 2: $k^* > k_0$. Construct dichotomous V in such a way that candidate a is in every voter's least-preferred subset and, with $a_1, a_2, \cdots, a_{m-1}$ the other $m-1$ candidates, a_i is a least-preferred candidate for the ith term in P. Then $a \notin \mathrm{Con}(V)$, and with $\alpha \in V(s)$ such that a is in the subset of k^* candidates assigned to each voter by α and a_i is not in the α-assigned subset of k^* candidates for the ith voter $(i = 1, \cdots, m-1)$, $F(\alpha) = \{a\}$. This completes the proof of the theorem.

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