Homework 4 (due Friday, October 18)

Exercise 1. (Buffon’s needle) We throw a needle of length $2\ell$ “randomly” on a floor on which are already drawn parallel vertical lines, separated by equal distance $2t$ - see Fig. 1a. This needle is called Buffon’s needle. Now, consider two Buffon’s needles with $\ell = t$. We know for a fact that the probability with which a single needle crosses one of the vertical lines is equal to $\frac{2}{\pi}$. Generally, we do not know anything about the relation of these two needles to each other. However, by adding some assumptions, one can create a coupling between the two needles. For each of the following couplings, compute the probability with which “both” needles cross a vertical line.

Coupling #1: The needles are independently thrown on the floor - Fig. 1b.

Coupling #2: The needles are welded at their ends to form a straight needle with length $4\ell = 4t$ - Fig. 1c.

Coupling #3: The needles are welded perpendicularly at their midpoints, yielding a cross - Fig. 1d.

![Figure 1: The same cup of coffee. Two times.](image)

Exercise 2. A complete graph $K_m$ with $m$ vertices is a graph with one edge between all pairs of vertices. We consider the homogeneous random walk $(X_n, n \geq 0)$ on $K_m$ defined by the transition probability $\mathbb{P}(X_{n+1} = j|X_n = i) = 1/(m-1)$ for all vertices $j \neq i$. We also denote by $\mu_i^{(n)} = \mathbb{P}(X_n = i)$, $i \in K_m$ the probability distribution of a random walk at time $n$ with initial condition $X_0 = i_0$ and by $\nu_j^{(n)} = \mathbb{P}(Y_n = j)$, $j \in K_m$ the probability distribution of another independent random walk with initial condition $Y_0 = j_0$. The two initial vertices are fixed once for all and distinct, $i_0 \neq j_0$. 

![Diagram of a complete graph and random walk](image)
We now define the following homogeneous Markov process \((X_n, Y_n), n \geq 0\) with state space \(K_m \times K_m\) and transition probabilities

\[
\mathbb{P}(X_{n+1} = i', Y_{n+1} = j' | X_n = i, Y_n = j) = \begin{cases} 
\frac{1}{(m-1)^2} & \text{if } i \neq j \text{ and } i' \neq i, j' \neq j, \\
\frac{1}{m-1} & \text{if } j = i \text{ and } j' = i' \neq i, \\
0 & \text{in all other cases.}
\end{cases}
\]

It is understood that this Markov process is conditioned on the initial condition \((X_0, Y_0) = (i_0, j_0)\).

a) Show that \((X_n, Y_n)\) is a coupling of the two probability distributions \(\mu^{(n)}\) and \(\nu^{(n)}\).

Hint: Recall the definition of coupling; you have to compute the marginals of \(\mathbb{P}(X_n = i, Y_n = j)\).

b) Consider the coalescence time \(T = \inf\{n \geq 1 \mid X_n = Y_n\}\) (a random variable). First show that for all \(n \geq 1:\)

\[
\mathbb{P}(T > n) = \left(1 - \frac{m-2}{(m-1)^2}\right)^n
\]

and deduce from there that for all \(n \geq 1:\)

\[
\mathbb{P}(T = n) = \frac{m-2}{(m-1)^2} \left(1 - \frac{m-2}{(m-1)^2}\right)^{n-1}
\]

Hint: rewrite the events \(\{T > n\}\) and \(\{T = n\}\) in terms of the \(X\)'s and \(Y\)'s.

c) Consider the total variation distance \(\|\mu^{(n)} - \nu^{(n)}\|_{TV}\). Use the formula in point 1 of exercise 2 to show that

\[
\|\mu^{(n)} - \nu^{(n)}\|_{TV} \leq e^{-n(m-2)/(m-1)^2}
\]

Hint: \(1 - x \leq e^{-x}\) for all \(x \geq 0\).

d) We remark that \(\pi_i = 1/m, i \in K_m\) is a stationary distribution for the random walk on the complete graph. Justify this remark. Use this remark and the previous bound to deduce

\[
\|\mu^{(n)} - \pi\|_{TV} \leq e^{-n(m-2)/(m-1)^2}
\]

e) What happens in the (very) particular case \(m = 2\)?

Exercise 3. a) Let \(\mu\) and \(\nu\) be two distributions on a state space \(S\) (i.e., \(\mu_i, \nu_i \geq 0\) for every \(i \in S\) and \(\sum_{i \in S} \mu_i = \sum_{i \in S} \nu_i = 1\)). Show that the following three definitions of the total variation distance between \(\mu\) and \(\nu\) are equivalent:

1. \(\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{i \in S} |\mu_i - \nu_i|\).

2. \(\|\mu - \nu\|_{TV} = \max_{A \subseteq S} |\mu(A) - \nu(A)|\), where \(\mu(A) = \sum_{i \in A} \mu_i\) and \(\nu(A) = \sum_{i \in A} \nu_i\).

3. \(\|\mu - \nu\|_{TV} = \frac{1}{2} \max_{\phi : S \rightarrow [-1,1]} |\mu(\phi) - \nu(\phi)|\), where \(\mu(\phi) = \sum_{i \in S} \mu_i \phi_i\) and \(\nu(\phi) = \sum_{i \in S} \nu_i \phi_i\).

Hint: The easiest way is to show that \(1 \leq 2 \leq 3 \leq 1\).

b) Show that \(\|\mu - \nu\|_{TV}\) is indeed a distance (i.e., that it is non-negative, that it is zero if and only if \(\mu = \nu\), that it is symmetric and that the triangle inequality is satisfied).