Solutions to Homework 14

Exercise 1. In order to use Chebyshev’s inequality, we simply write for \( t < \mathbb{E}(X_1) = \frac{1}{\lambda} \):

\[
P(\{S_n \leq nt\}) = P(\{-S_n \geq -nt\}) \leq \frac{\mathbb{E}(\exp(-sS_n))}{\exp(-snt)}
\]
valid for any \( s \geq 0 \). The rest follows closely the analysis done in Homework 11, ex. 1:

\[
P(\{S_n \leq nt\}) \leq \exp(snt) \mathbb{E}(\exp(-sX_1))^n
\]
and \( \mathbb{E}(\exp(-sX_1)) = \int_0^\infty e^{-sx} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda + s} \), so

\[
P(\{S_n \leq nt\}) \leq \exp(n(st - \log(s + \lambda) + \log(\lambda)))
\]
which is valid for every \( s \geq 0 \), therefore

\[
P(\{S_n \leq nt\}) \leq \exp\left(n \min_{s \geq 0} (st - \log(s + \lambda) + \log(\lambda))\right)
\]
The minimum is reached in \( s^* = -\lambda + \frac{1}{t} > 0 \) (as \( t < \frac{1}{\lambda} \) by assumption), which gives finally

\[
P(\{S_n \leq nt\}) \leq \exp(n(-\lambda t + 1 + \log(\lambda t))) = \exp(-n(\lambda t - 1 - \log(\lambda t)))
\]
and we check that \( \lambda t - 1 - \log(\lambda t) > 0 \), as \( \log(x) < x - 1 \) for \( x < 1 \). The expression found is therefore the same as that obtained for \( P(\{S_n \geq nt\}) \), but the function has a quite different behaviour, as we can observe on the figure below (where we set \( \lambda = 1 \)):

In particular, when \( t \) approaches zero, the function goes to \(+\infty\), matching with the fact that \( P(\{S_n \leq 0\}) = 0 \).
Exercise 2. a) We have
\[ \mathbb{P}(|M_n| > nt) = \mathbb{P}(M_n > nt) + \mathbb{P}(M_n < -nt) = 2 \mathbb{P}(M_n > nt) \]
because of the symmetry of the distribution of \( M_n \). Using next Chebyshev’s inequality with \( \psi(x) = \exp(sx) \) and \( s \geq 0 \), we obtain
\[ \mathbb{P}(M_n > nt) \leq \frac{\mathbb{E}(\exp(sM_n))}{\exp(snt)} = \exp(-snt) \prod_{j=1}^{n} \mathbb{E}(\exp(sX_j)) \]
Using the hint, we further obtain
\[ \mathbb{P}(M_n > nt) \leq \exp(-snt) \prod_{j=1}^{n} \exp(s^2 a_j^2/2) = \exp\left(-n^2 t^2 s^2 + \frac{s^2}{2} \sum_{j=1}^{n} a_j^2\right) \]
Let \( D_n = \sum_{j=1}^{n} a_j^2 \). Optimizing over \( s \geq 0 \), we find \( s^* = nt/D_n \), so that finally
\[ \mathbb{P}(M_n > nt) \leq 2 \mathbb{P}(M_n > nt) \leq 2 \exp\left(-\frac{n^2 t^2}{2D_n}\right) \]
b) Observe that \( D_n \to \frac{\varepsilon^2}{n} \). Therefore
\[ \mathbb{P}(|M_n| > nt) \leq 2 \exp(-c(nt)^2) \]
for some constant \( c > 0 \) as \( n \) gets large. For \( \alpha > 0 \) and \( \varepsilon > 0 \), we therefore conclude that
\[ \mathbb{P}(|M_n| > n^\alpha \varepsilon) \leq 2 \exp(-c n^{2\alpha} \varepsilon^2) \]
The Borel-Cantelli lemma then implies that \( \mathbb{P}(\{|M_n| \leq n^\alpha \varepsilon \text{ infinitely often}\}) = 0 \) for every \( \varepsilon > 0 \), proving that \( \frac{M_n}{n^\alpha} \to 0 \) almost surely.

c) We have \( \mathbb{E}(M_n^2) = \sum_{j=0}^{n} \mathbb{E}(X_j^2) = \sum_{j=1}^{n} a_j^2 \), so by the martingale convergence theorem (first version), if
\[ \sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) = \sum_{n \in \mathbb{N}} a_n^2 < +\infty \]
then the two conclusions in the problem set hold.

d) By the second version of the martingale convergence theorem, we only need the weaker condition
\[ \sup_{n \in \mathbb{N}} \mathbb{E}(|M_n|) < +\infty \] 
(1)
in order for almost sure convergence to take place. Watch out however that in general,
\[ \sup_{n \in \mathbb{N}} \mathbb{E}(|M_n|) \neq \sum_{n \in \mathbb{N}} a_n \] !!!
Asking indeed that \( \sum_{n \in \mathbb{N}} a_n < +\infty \) is a stronger condition than \( \sum_{n \in \mathbb{N}} a_n^2 < +\infty \), not a weaker one. What holds true is that \( \mathbb{E}(|M_n|) \leq \sum_{j=0}^{n} \mathbb{E}(|X_j|) = \sum_{j=1}^{n} a_j \), so that (1) is satisfied if \( \sum_{n \in \mathbb{N}} a_n < +\infty \). But expressing (1) in terms of the numbers \( a_n \) is not an easy task in general.
e) In this case, \( \sum_{n \in \mathbb{N}} a_n^2 = \sum_{n \in \mathbb{N}} \frac{1}{n^2} < +\infty \), so both \( \sup_{n \in \mathbb{N}} \mathbb{E}(M_n^2) < +\infty \) and \( \sup_{n \in \mathbb{N}} \mathbb{E}(|M_n|) < +\infty \) hold, as well as the two conclusions of the first version of the martingale convergence theorem.