Homework 14

Exercise 1. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. In Homework 11, ex. 1, the following upper bound was found (using the large deviations principle):

$$
\mathbb{P}(\{S_n \geq nt\}) \leq \exp(-n(\lambda t - 1 - \log(\lambda t))) \quad \text{for } t > \mathbb{E}(X_1) = \frac{1}{\lambda}
$$

As $\log(x) < x - 1$ for $x > 1$, one checks indeed that $\lambda t - 1 - \log(\lambda t) > 0$ for $t > \frac{1}{\lambda}$, proving the exponential decay (in $n$) of the probability.

Here, we ask you to find a corresponding upper bound on $\mathbb{P}(\{S_n \leq nt\})$, for $t < \mathbb{E}(X_1) = \frac{1}{\lambda}$.

Note: Because the random variables $X_i$ are non-negative, there is an inherent asymmetry in the problem.

Exercise 2. Let $(a_n, n \geq 1)$ be a decreasing sequence of positive numbers and $(X_n, n \in \mathbb{N})$ be a sequence of independent random variables such that

$$
\mathbb{P}(\{X_n = +a_n\}) = \mathbb{P}(\{X_n = -a_n\}) = \frac{1}{2} \quad \forall n \geq 1
$$

Let also $(M_n, n \in \mathbb{N})$ be the process defined as $M_0 = 0$ and $M_n = \sum_{j=1}^{n} X_j$ for $n \geq 1$.

a) Find a tight upper bound on

$$
\mathbb{P}(\{|M_n| \geq nt\})
$$

Hint: For this, you may use the following (which is a small adaptation of what we have seen in the course): if $X$ is a random variable taking values $+a$ and $-a$ with probability $1/2$, then

$$
\mathbb{E}(\exp(sX)) = \frac{e^{sa} + e^{-sa}}{2} = \cosh(sa) \leq \exp(s^2 a^2 / 2) \quad \forall s \in \mathbb{R}
$$

b) Application: assume now that $a_n = \frac{1}{n}$ for $n \geq 1$. Find the least value of $\alpha > 0$ for which the upper bound found in a) allows to conclude that

$$
\frac{M_n}{n^{\alpha}} \to 0 \quad \text{almost surely}
$$

for $n \to \infty$.

c) Back to the general case: Considering the process $M$ as a martingale (and letting $(\mathcal{F}_n, n \in \mathbb{N})$ be its natural filtration), what condition on the decreasing sequence $(a_n, n \geq 1)$ ensures that there exists a random variable $M_\infty$ such that

$$
M_n \to M_\infty \quad \text{almost surely} \quad \text{and} \quad \mathbb{E}(M_\infty | \mathcal{F}_n) = M_n \quad \forall n \in \mathbb{N}?
$$

d) Is there a simple condition on the decreasing sequence $(a_n, n \geq 1)$ ensuring only that

$$
M_n \to M_\infty \quad \text{almost surely}?
$$

[Please pay attention: this question is a (big) trap!]

e) Which of the above two conditions is satisfied by the sequence $a_n = 1/n$?