Exercise 1. Let $X$ be a random variable such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Z \sim \mathcal{N}(0, 1)$ be independent of $X$. Let also $a > 0$ and $Y = aX + Z$. We propose below four possible estimators of the variable $X$ given the noisy observation $Y$:

$$
\hat{X}_1 = \frac{Y}{a} \quad \hat{X}_2 = \frac{aY}{a^2 + 1} \quad \hat{X}_3 = \text{sign}(aY) \quad \hat{X}_4 = \tanh(aY)
$$

a) Which estimator among these four minimizes the mean squared error (MSE) $\mathbb{E}((\hat{X} - X)^2)$?

In order to answer the question, draw on the same graph the four curves representing the MSE as a function of $a > 0$. For this, you may use either the exact mathematical expression of the MSE or the one obtained via Monte-Carlo simulations.

b) Provide a theoretical justification for your conclusion.

c) For which of the four estimators above does it hold that $\mathbb{E}((\hat{X} - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\hat{X}^2)$?

Exercise 2. Part I. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. centered and bounded random variables; let $(\mathcal{F}_n, n \geq 1)$ be the filtration defined as $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$. Among the following processes $(Y_n, n \geq 1)$, which are martingales with respect to $(\mathcal{F}_n, n \geq 1)$? (just a short justification suffices here)

a) $Y_n = X_n$, $n \geq 1$.

b) $Y_1 = X_1$, $Y_{n+1} = aY_n + X_{n+1}$, $n \geq 1$ ($a > 0$ fixed).

c) $Y_1 = X_1$, $Y_{n+1} = X_n + X_{n+1}$, $n \geq 1$.

d) $Y_n = \max(X_1, \ldots, X_n)$, $n \geq 1$.

e) $Y_1 = X_1$, $Y_n = \sum_{i=1}^{n} (X_1 + \ldots + X_{i-1})X_i$, $n \geq 1$.

Part II. Let now $(S_n, n \in \mathbb{N})$ be the symmetric random walk and $(\mathcal{F}_n, n \in \mathbb{N})$ be its natural filtration. Among the following random times, which are stopping times with respect to $(\mathcal{F}_n, n \in \mathbb{N})$? which are bounded? (no justification required here)

a) $T = \sup\{n \geq 0 : S_n \geq a\}$ ($a > 0$ is fixed)

b) $T = \inf\{n \geq 1 : S_n = \max_{0 \leq k \leq n} S_k\}$

c) $T = \inf\{n \geq 0 : S_n = \max_{0 \leq m \leq N} S_m\}$ ($N \geq 1$ is fixed)

d) $T = \inf\{n \geq 0 : S_n \geq a \text{ or } n \geq N\}$ ($a > 0$ and $N \geq 1$ are fixed)

e) $T = \inf\{n \geq 0 : |S_n| \geq a\}$ ($a > 0$ is fixed)
Exercise 3.  

a) Let \((M_n, n \in \mathbb{N})\) be an *increasing* martingale, that is, \(M_{n+1} \geq M_n\) a.s. for all \(n \in \mathbb{N}\). Show that \(M_n = M_0\) a.s., for all \(n \in \mathbb{N}\).

b) Let \((M_n, n \in \mathbb{N})\) be a square-integrable martingale such that \((M_n^2, n \in \mathbb{N})\) is also a martingale. Show that \(M_n = M_0\) a.s., for all \(n \in \mathbb{N}\).

Exercise 4.  

Let \((M_n, n \in \mathbb{N})\) be a submartingale with respect to a filtration \((\mathcal{F}_n, n \in \mathbb{N})\) and \(\varphi : \mathbb{R} \to \mathbb{R}\) be a Borel-measurable and convex function such that \(E(|\varphi(M_n)|) < +\infty, \forall n \in \mathbb{N}\).

a) What additional property of \(\varphi\) ensures that the process \((\varphi(M_n), n \in \mathbb{N})\) is also a submartingale?

b) In particular, which of the following processes is ensured to be a submartingale: \((M_n^2, n \in \mathbb{N})\) and/or \((\exp(M_n), n \in \mathbb{N})\)?

Let \((X_n, n \geq 1)\) be a sequence of i.i.d. random variables such \(P(\{X_1 = +1\}) = P(\{X_1 = -1\}) = \frac{1}{2}\); let \(S_0 = 0\) and \(S_n = X_1 + \ldots + X_n\) for \(n \geq 1\); finally, let \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\) for \(n \geq 1\).

c) Show that \((S_n^2 - n, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)

d) For which value of \(c > 0\) is the process \(\left(\frac{\exp(S_n)}{c^n}, n \in \mathbb{N}\right)\) a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)?

Assume now that \(P(\{X_1 = +1\}) = p = 1 - P(\{X_1 = -1\})\) for some \(0 < p < 1\) with \(p \neq \frac{1}{2}\).

e) Does there exist a number \(c > 0\) such that the process \((S_n^2 - cn, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? If yes, compute the value of \(c\); otherwise, justify why it is not the case.

f) Does there exist a number \(c > 0\) such that the process \(\left(\frac{\exp(S_n)}{c^n}, n \in \mathbb{N}\right)\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? If yes, compute the value of \(c\); otherwise, justify why it is not the case.