Solutions to Homework 11

Exercise 1. We use the large deviations principle to find a tight upper bound. Before this, we need to check that the moment generating function $\mathbb{E}(e^{sX_1})$ is finite in a proper neighborhood of $s = 0$:

$$\mathbb{E}(e^{sX_1}) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda - s}, \quad \text{for } s < \lambda$$

Therefore, by applying the large deviations principle, we obtain for $t > 1/\lambda$:

$$\mathbb{P}(\{S_n > nt\}) \leq \exp(-n \Lambda^*(t))$$

where

$$\Lambda^*(t) = \max_{s \in \mathbb{R}} \left\{ st - \log \left( \frac{\lambda}{\lambda - s} \right) \right\}$$

By taking the derivative of $st - \log \left( \frac{\lambda}{\lambda - s} \right)$ with respect to $s$ and setting it equal to zero, we obtain that $\Lambda^*(t)$ is maximum at $s^* = \lambda - \frac{1}{t}$. Hence,

$$\mathbb{P}(\{S_n > nt\}) \leq \exp(-n (\lambda t - 1 - \log(\lambda t)))$$

Exercise 2. Here is the plot of the 3 functions $f, g, h$ for $n = 100$:

It gets difficult to compute the function $f$ numerically for large values of $n$, as events become more and more rare.

Exercise 3. a) Use part (ii) of the definition with $U \equiv 1$ ($U$ is $\mathcal{G}$-measurable and bounded).

b) (i) $\mathbb{E}(X)$ is constant and therefore $\mathcal{G}$-measurable; (ii) Let $U$ be $\mathcal{G}$-measurable and bounded:

$$\mathbb{E}(XU) = \mathbb{E}(X) \mathbb{E}(U) = \mathbb{E}(\mathbb{E}(X)U)$$

(using the independence of $X$ and $U$ and the linearity of expectation).

c) (i) $X$ is $\mathcal{G}$-measurable by assumption; (ii) Let $U$ be $\mathcal{G}$-measurable and bounded: $\mathbb{E}(XU) = \mathbb{E}(XU)$!

d) (i) $\mathbb{E}(X|\mathcal{G}) Y$ is $\mathcal{G}$-measurable; (ii) Let $U$ be $\mathcal{G}$-measurable and bounded: $\mathbb{E}(XYU) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}) YU)$.

e) Let us first check the left-hand side equality: $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable, therefore $\mathcal{G}$-measurable, so one can apply property c). For the right-hand side equality, one has: (i) $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable; (ii) Let $U$ be $\mathcal{H}$-measurable and bounded:

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}) U) = \mathbb{E}(\mathbb{E}(XU|\mathcal{G})) = \mathbb{E}(XU) = \mathbb{E}(\mathbb{E}(X|\mathcal{H}) U)$$

using d), a) and the definition of $\mathbb{E}(X|\mathcal{H})$. 