Exercise 1. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $X_1$ admits the following pdf:

$$p_{X_1}(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Let also $S_n = X_1 + \ldots + X_n$. Using the large deviations principle, find a tight upper bound on

$$\mathbb{P}(\{S_n \geq nt\}) \quad \text{for} \quad t > \mathbb{E}(X_1) = \frac{1}{\lambda}$$

Exercise 2. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(\{X_1 = 1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$$

Let also $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. For a fixed value of $n$, draw on the same graph the following functions:

$$f(t) = -\frac{1}{n} \log \mathbb{P}(\{S_n > nt\})$$
$$g(t) = \Lambda^*(t) = \max_{s \in \mathbb{R}}(st - \Lambda(s)) \quad \text{where} \quad \Lambda(s) = \log \mathbb{E}(e^{sX_1})$$
$$h(t) = t^2/2$$

NB: On these plots, $t \in [0, +1]$. In order to draw the function $f(t)$, you should use Monte-Carlo simulation, that is, draw i.i.d. samples $X_1^{(m)}, \ldots, X_n^{(m)}$ for $m = 1, \ldots, M$ (with $M$ reasonably large) and approximate $f(t)$ as

$$f(t) \simeq -\frac{1}{n} \log \left( \frac{1}{M} \#\{1 \leq m \leq M : S_n^{(m)} > nt\} \right)$$

As you will see, considering even moderate values of $n$ requires considering quite large values of $M$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ be a square-integrable random variable defined on this space and let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$. Relying only on the definition of conditional expectation (choose your favorite!), show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If $X$ is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If $X$ is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If $Y$ is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})Y$ a.s.

e) If $\mathcal{H}$ is a sub-$\sigma$-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.