Homework 9

Exercise 1. Let \((G_n, n \geq 1)\) be a sequence of Gaussian random variables such that \(G_n \sim \mathcal{N}(\mu_n, \sigma_n^2)\) for \(n \geq 1\).

a) Show, using characteristic functions, that if \(\mu_n \xrightarrow{n\to\infty} \mu \in \mathbb{R}\) and \(\sigma_n^2 \xrightarrow{n\to\infty} \sigma^2 \geq 0\), then

\[G_n \xrightarrow{d} n\to\infty G \sim \mathcal{N}(\mu, \sigma^2)\]

Note. This by the way provides a formal justification for the answer to the bonus question in the midterm (exercise 3.d).

b) Show, still using characteristic functions, that if \(\sigma_n^2 \xrightarrow{n\to\infty} +\infty\), then the sequence \((G_n, n \geq 1)\) does not converge in distribution, irrespective of the sequence \((\mu_n, n \geq 1)\).

Let now \((X_n, n \geq 1)\) be a sequence of independent random variables such that

\[
P\left(\{X_n = +1/\sqrt{n}\}\right) = P\left(\{X_n = -1/\sqrt{n}\}\right) = \frac{1}{2}
\]

For \(n \geq 1\), let also

\[Y_n = X_1 + \ldots + X_n \quad \text{and} \quad Z_n = X_{n+1} + \ldots + X_{2n}\]

c) Either following Lindeberg’s approach or using characteristic functions (or both!), show that the sequence of random variables \((Z_n, n \geq 1)\) converges in distribution towards a Gaussian random variable \(Z\). Compute the variance of this random variable.

Hints: - If \(f : \mathbb{R}_+ \to \mathbb{R}_+\) is a decreasing function, then it holds that

\[
\sum_{k=n_1+1}^{n_2} f(k) \simeq \int_{n_1}^{n_2} dx \ f(x) \ \text{as} \ n_1, n_2 \to \infty
\]

- If you follow Lindeberg’s approach, you will also need part a) here. If you use characteristic functions, you will need to perform some approximations via Taylor expansions.

d) Show that the sequence of random variables \((Y_n, n \geq 1)\) does not converge in distribution.

Note. There is an “intuitive reasoning” leading to the conclusion here, but there is also a formal one, that actually uses part c)! (Hint: Consider the subsequence \((Y_{2^k}, k \geq 1)\).)

Exercise 2. (Why it is not a good idea to play at roulette too many times)

On a classical roulette game with 38 numbers (including the 0 and the 00), a player bets uniquely on red, 361 times in a row. At each turn, he bets exactly one franc (he therefore wins one franc if red comes out and loses one franc if this is not the case). Assuming that the roulette wheel is balanced and that the turns are independent from each other, give a rough estimate of:

a) the average player’s fortune at the end of the 361 games;

b) the probability that he has actually won some money.

NB: Remember that the numbers 0 and 00 are neither red nor black on a classical roulette.