Exercise 1. Let \((X_n, n \geq 1)\) be a sequence of independent random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\) and such that \(\exists \varepsilon_0 > 0\) with \(P(|X_n| > \varepsilon) = \frac{1}{n}\), for every \(n \geq 1\) and \(0 < \varepsilon < \varepsilon_0\).

Show that a) \(X_n \xrightarrow{P} 0\) but b) \(X_n \nrightarrow 0\) almost surely.

Method for proving b):
1. Show first that for every \(N \geq 1\) and \(\varepsilon > 0\), we have \(P\left(\bigcap_{n \geq N} \{|X_n| \leq \varepsilon\}\right) = 0\).

Hint. Use the inequality \(1 - x \leq e^{-x}\), valid for all \(x \in \mathbb{R}\).

2. Remember that \(X_n \rightarrow 0\) almost surely if and only if \(\forall \varepsilon > 0, \ P\left(\bigcup_{N \geq 1} \bigcap_{n \geq N} \{|X_n| \leq \varepsilon\}\right) = 1\).

Apply (or not!) the above result to answer the following questions:

- c) Let \(Y \sim U([0, 1])\) and \(X_n = \sqrt{n} 1_{Y \leq 1/n}\) for \(n \geq 1\).
  Does the sequence of random variables \((X_n, n \geq 1)\) converge in \(L^2\)? in probability? almost surely?

- d) Let \(Y_n\) be i.i.d. \(\sim U([0, 1])\) random variables and \(X_n = \sqrt{n} 1_{Y_n \leq 1/n}\) for \(n \geq 1\).
  Does the sequence of random variables \((X_n, n \geq 1)\) converge in \(L^2\)? in probability? almost surely?

Exercise 2. Let \((X_n, n \geq 1)\) be a sequence of i.i.d. non-negative random variables defined on a common probability space \((\Omega, \mathcal{F}, P)\) and such that \(E(|\log(X_1)|) < \infty\). Let also \((Y_n, n \geq 1)\) be the sequence defined as

\[
Y_n = \left(\prod_{j=1}^{n} X_j\right)^{1/n}, \quad n \geq 1
\]

a) Show that there exists a constant \(\mu > 0\) such that \(Y_n \xrightarrow{n \to \infty} \mu\) almost surely.

b) Compute the value of \(\mu\) in the case where \(X_n = \exp(N_n)\) and \((N_n, n \geq 1)\) are i.i.d. \(\sim N(0, 1)\) random variables.

c) In this case, look for the tightest possible upper bound on \(P(\{Y_n > t\})\) for \(n \geq 1\) fixed and \(t > \mu\).

Hint. You have two options here. One is to use Chebyshev’s inequality with the function \(\psi(x) = x^p\) and \(p > 0\) (and then optimize over \(p\)) in order to upperbound

\[
P(\{Y_n > t\}) = P\left(\prod_{j=1}^{n} X_j > t^n\right)
\]

for \(t > \mu\). The other option is left to your imagination...
Exercise 3. (extended law of large numbers)
Let \((\mu_n, n \geq 1)\) be a sequence of real numbers such that
\[
\lim_{n \to \infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \mu \in \mathbb{R}
\]
Let \((X_n, n \geq 1)\) be a sequence of square-integrable random variables such that
\[
\mathbb{E}(X_n) = \mu_n, \quad \forall n \geq 1 \quad \text{and} \quad \text{Cov}(X_n, X_m) \leq C_1 \exp(-C_2 |m - n|), \quad \forall m, n \geq 1.
\]
for some constants \(C_1, C_2 > 0\) (the random variables \(X_n\) are said to be weakly correlated). Let finally \(S_n = X_1 + \ldots + X_n\).

a) Show that
\[
\frac{S_n}{n} \xrightarrow{p} \mu
\]
b) Is it also true that
\[
\frac{S_n}{n} \xrightarrow{a.s.} \mu \quad \text{almost surely?}
\]
In order to check this, you need to go through the proof of the strong law of large numbers made in class. Does that proof need the fact that the random variables \(X_n\) are independent?

c) Application: Let \((Z_n, n \geq 1)\) be a sequence of i.i.d. \(\mathcal{N}(0, 1)\) random variables, \(x, a \in \mathbb{R}\) and \((X_n, n \geq 1)\) be the sequence of random variables defined recursively as
\[
X_1 = x, \quad X_{n+1} = aX_n + Z_{n+1}, \quad n \geq 1
\]
For what values of \(x, a \in \mathbb{R}\) does the sequence \((X_n, n \geq 1)\) satisfy the assumptions made in a)? Compute \(\mu\) in this case.