Solutions to Graded Homework 6

Exercise 1. a) use $\psi(x) = x^2$ and $\psi(x) = x^2 + \sigma^2$ respectively.

b) By the hint, $P(\{X \geq a\}) \leq \frac{\sigma^2 + b^2}{(a+b)^2} = g(b)$. $g$ has a minimum in $b = \frac{\sigma^2}{a}$ and at this point, $g(b) = \frac{\sigma^2}{a^2 + \sigma^2}$.

Exercise 2. a) Using the fact that $X$ takes values in $[0,1]$ and Chebyshev’s inequality, we obtain

$$P(\left\{ X \leq \frac{1}{4} \right\}) = P(\left\{ 1 - X \geq \frac{3}{4} \right\}) \leq \frac{E(1-X)}{3/4} \leq \frac{1/2}{3/4} = \frac{2}{3}$$

b) The only random variable satisfying all the conditions is the random variable $X$ such that

$$P(\{X = \frac{1}{4}\}) = \frac{2}{3} \quad \text{and} \quad P(\{X = 1\}) = \frac{1}{3}$$

c) 1. There are multiple ways to derive this inequality, all using Chebyshev’s inequality. For example:

$$P(\{X = 0\}) = P(\{-X = 0\}) \leq P(\{E(X) - X \geq E(X)\}) \leq \frac{E((E(X) - X)^2)}{E(X)^2} = \frac{\text{Var}(X)}{E(X)^2}$$

2 and 3. Using the hint, we obtain

$$E(X 1_{\{X > 0\}})^2 \leq E(X^2) P(\{X > 0\})$$

from which we deduce that

2. $P(\{X = 0\}) = 1 - P(\{X > 0\}) \leq 1 - \frac{E(X 1_{\{X > 0\}})^2}{E(X^2)} = 1 - \frac{E(X)^2}{E(X^2)} = \frac{\text{Var}(X)}{E(X)^2}$

and

3. $P(\{X > 0\}) \geq \frac{E(X 1_{\{X > 0\}})^2}{E(X^2)} = \frac{E(X)^2}{E(X^2)}$

d) If $X$ is a Poisson random variable with parameter $\lambda$, then $P(\{X = 0\}) = e^{-\lambda}$, $P(\{X > 0\}) = 1 - e^{-\lambda}$, $E(X) = \text{Var}(X) = \lambda$ and $E(X^2) = \lambda(\lambda + 1)$. So we check indeed that

2. $e^{-\lambda} \leq \frac{1}{\lambda + 1}$ (true as $e^\lambda \geq \lambda + 1, \forall \lambda \in \mathbb{R}$), implying 1. $e^{-\lambda} \leq \frac{1}{\lambda}$

3. $1 - e^{-\lambda} \geq \frac{\lambda}{\lambda + 1}$ which is equivalent to $e^{-\lambda} \leq \frac{1}{\lambda + 1}$

Exercise 3. a) Saying that $X$ is a bounded and non-negative random variable amounts to saying that there exists $C > 0$ such that $0 \leq X(\omega) \leq C$ for all $\omega \in \Omega$, so $E(X^k) \leq E(C^k) = C^k < +\infty$. Another way to phrase this is to say that as $X$ is bounded, $X^k$ also is, so $X^k$ is integrable.
b) Using Cauchy-Schwarz' inequality, we obtain directly the result.
c) Using Jensen’s inequality and the fact that the map \( x \mapsto x^{\ell/k} \) is convex for \( \ell \geq k \), we obtain
\[
\mathbb{E}(X^\ell) = \mathbb{E}\left( (X^k)^{\ell/k} \right) \geq (\mathbb{E}(X^k))^{\ell/k}
\]
which leads to the result.
d) As \( 0 \leq X(\omega) \leq 1 \) for all \( \omega \in \Omega \), we obtain for \( \ell \geq k \):
\[
\mathbb{E}(X^\ell) \leq \mathbb{E}(X^k \cdot 1^{\ell-k}) = \mathbb{E}(X^k) \leq 1^k = 1
\]
e) In view of the previous inequalities, the sequences 2 and 4 are not the moments \( m_k = \mathbb{E}(X^k) \) of a random variable \( X \) taking values in the interval \([0, 1]\), because:

2. \( m_k = 1 - \frac{1}{k+1} \) is increasing (violates condition d)

4. \( m_k = \frac{1}{k^e} \) violates condition c: \( m_{\ell}^{1/\ell} = \frac{1}{\ell} < m_k^{1/k} = \frac{1}{k} \) for \( \ell > k \geq 1 \).

One can check on the other hand that the sequences 1 and 3 satisfy all the required inequalities. The sequence 1 corresponds to \( X \sim \mathcal{U}([0, 1]) \) and the sequence 3 corresponds to \( X(\omega) = \frac{1}{2} \forall \omega \in \Omega \).

Note. It is an important fact that the sequence of moments of a bounded random variable completely determines the distribution of this random variable (the non-negativity assumption is not needed; it was just assumed for convenience here).

Exercise 4. a) See the notebook. For different realizations (i.e., different \( \omega \)'s), the sequence \( (Y_n(\omega), n \geq 1) \) converges to different values, but it always converges!

b) We show below that for every \( \omega \in \Omega \), \( (Y_n(\omega), n \geq 1) \) is a Cauchy sequence, implying the existence of a limit \( Y(\omega) \). For \( m > n \geq 1 \), we have
\[
|Y_m(\omega) - Y_n(\omega)| = \left| \sum_{j=n+1}^{m} X_j(\omega) \frac{1}{2^j} \right| \leq \sum_{j=n+1}^{m} \frac{1}{2^j} \leq \frac{1}{2^n} \xrightarrow{n,m \to \infty} 0
\]

Note that the limit may be written as the infinite series \( Y(\omega) = \sum_{j=1}^{\infty} X_j(\omega) \cdot \frac{1}{2^j} \).

c) See the notebook. The graphs shows an estimate of \( \mathbb{E}((Y_n - Y)^2) \) as a function of \( n \).

d) Let us compute:
\[
\mathbb{E}((Y_n - Y)^2) = \mathbb{E}\left( \left( \sum_{j \geq n+1} \frac{X_j(\omega)}{2^j} \right)^2 \right) = \sum_{j,k \geq n+1} \frac{\mathbb{E}(X_j X_k)}{2^{j+k}} \leq \sum_{j,k \geq n+1} \frac{1}{2^{j+k}} = \left( \sum_{j \geq n+1} \frac{1}{2^j} \right)^2 = \frac{1}{2^{2n}} \xrightarrow{n \to \infty} 0
\]

e) See the notebook. The histogram shows that the distribution of \( Y \) is uniform on \([0, 1]\). This can be inferred from the fact that every bit \( X_j \) of the binary decomposition of \( Y \) is chosen uniformly at random.