Exercise 1

Consider a Hopfield network composed of 9 neurons. Each neuron has connections to all other neurons.

1.1 How many connections are there in total? Choose the appropriate weights for the prototype pattern given in figure 1.

![Figure 1: Prototype pattern. Black corresponds to \( S = +1 \).](image)

Now keeping the learned weights fixed, present a pattern \( S_i(t = 0) \) and let it evolve according to:

\[
S_i(t + 1) = \text{sign} \left( \sum_j w_{ij} S_j(t) \right)
\]  

(1)

Suppose the initial state is again the swiss cross above but with one bit (neuron) flipped. Will the dynamics correct it?

1.2 Suppose that \( N \) bits are flipped. Will the dynamics correct them?
Exercise 2: Associative memory

Consider a Hopfield network with a continuous state variable $S_i(t) \in \mathbb{R}$. Assume that the network has stored 4 patterns

$$p^1 = \{p_1^1, \ldots, p_N^1\} \quad \cdots \quad p^4 = \{p_1^4, \ldots, p_N^4\}$$

that are orthogonal, i.e., $\frac{1}{N} \sum_{i=1}^{N} p^\mu_i p^\nu_i = \delta^\mu\nu$, where $\delta^\mu\nu$ is the Kronecker symbol

$$\delta^\mu\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

You present the network with an activity pattern that has overlap\(^1\) with $p^3$ only (no overlap with other memories). The activity dynamics is given by

$$S_i(t+1) = g \left( \sum_j w_{ij} S_j(t) \right)$$

2.1 Calculate the change of the overlap with pattern 3 in one time step, i.e. calculate $m^3(t+1)$ as a function of $m^3(t)$. Moreover, $g(\cdot)$ is an odd function: $g(-x) = -g(x)$

*Hint:* Follow the derivations shown in class (and in the book Neuronal Dynamics, chapter 17.2): Use the definitions of the overlap $m^3(t)$ and the weights $w_{ij}$ to express $S_i(t+1)$ (eq. 3) as a function of the overlap. Then, using $S_i(t+1)$ compute the overlap $m^3(t+1)$. Keep in mind that the state of each neuron always takes one of two values: $p_i \in \{-1, 1\}$.

2.2 Use this to discuss the evolution of the overlap over several time steps

- when $g$ is the sign-function
- when $g$ is an odd and monotonically increasing function mapping the real line onto $[-1; 1]$. As an example, consider $g(x) = \tanh(\beta x)$ with some real, positive parameter $\beta$. Think about the effect of changing $\beta$ (sometimes called 'inverse temperature') and discuss the cases $\beta < 1$, $\beta > 1$ and $\beta \to \infty$.

Exercise 3: Probability of error in the Hopfield model

3.1 Consider a Hopfield network of $N$ neurons ($N = 10'000$) storing $P$ random prototypes $p^\mu$ and the following dynamics:

$$S_i(t+1) = \text{sign} \left( \sum_j w_{ij} S_j(t) \right)$$

Given the initial activation set to pattern 1, i.e. $S_i(t = 0) = p_1^1$, show that

$$S_i(t = 1) = p_1^1 \text{sign}(1 + \sum_{\mu \neq 1}^{P} \sum_{j}^{N} \frac{1}{N} p_1^1 p_1^\mu p_j^1 p_j^\mu).$$

\(^1\)by “having overlap with prototype $\mu$” we mean with “having non-zero scalar product with $p^\mu$"
**Hint:** Start with the dynamics equation 4. Use the definition of the weights $w_{ij}$ to express the update in terms of the patterns.

**Hint:** You can always multiply a term with 1. In particular, with $1 = p_i^1 p_i^1$.

**3.2** In equation 5, formulate the condition for which $S_i$ will change its state. That is, $S_i(t = 1) \neq S_i(t = 0)$.

**3.3** Using the analogy for the sum as a random walk, show that the term $\sum_{\mu \neq 1}^{P} \sum_{j}^{N} \frac{1}{N} p_i^\mu p_i^j p_i^1 p_i^j$ can be approximated by a Gaussian random variable, $N(0, (P - 1)/N)$. 
**Hint:** Specify mean and variance of the distribution of the random variable $X = p_i^1 p_i^j p_i^1 p_i^j$. Then use the central limit theorem to approximate the sum by a Gaussian.

**3.4** Show that the probability that a given neuron $i$ will flip ($S_i(t = 1) \neq S_i(t = 0)$) is given by

$$P_{\text{error}} = \frac{1}{2} \left[1 - \text{erf}\left(\sqrt{\frac{N}{2(P - 1)}}\right)\right]$$

where erf is the error function, defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x'^2} dx'^2.$$  

**3.5** How many random patterns can you store, if you accept on average at most 1 bit to be wrong? Consider erf(2.6) = 0.9998.

**3.6** In many real application, patterns to be stored are not totally random and have substantial overlap. Rewrite the retrieval equation 5 as a function of overlap terms, $m^{\mu\nu} = \frac{1}{N} \sum_{i} p_i^\mu p_i^\nu$.

**3.7** Assume that the overlap between different patterns is 0.1 for all pairs. How many patterns can you store now, allowing on average only one wrong bit?

![Figure 2: Error probability: $P(x \leq -1)$](image-url)