**Homework 3**

**Exercise 1.** Let \( n \geq 1, \Omega = \{1, 2, \ldots, n\}, \mathcal{F} = \mathcal{P}(\Omega) \) and \( \mathbb{P} \) be the probability measure on \((\Omega, \mathcal{F})\) defined by \( \mathbb{P}(\{\omega\}) = \frac{1}{n} \) on the singletons and extended by additivity to all subsets of \( \Omega \).

a) Consider first \( n = 4 \). Find three subsets \( A_1, A_2, A_3 \subset \Omega \) such that
\[
\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{but} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)
\]
b) Consider now \( n = 6 \). Find three subsets \( A_1, A_2, A_3 \subset \Omega \) such that
\[
\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3) \quad \text{but} \quad \exists j \neq k \text{ such that } \mathbb{P}(A_j \cap A_k) \neq \mathbb{P}(A_j) \cdot \mathbb{P}(A_k)
\]
c) Consider finally a generic probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and three events \( A_1, A_2, A_3 \in \mathcal{F} \) such that
\[
\mathbb{P}(A_j \cap A_k) = \mathbb{P}(A_j) \cdot \mathbb{P}(A_k) \quad \forall j \neq k \quad \text{and} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)
\]
Show that \( A_1, A_2, A_3 \) are independent according to the definition given in the course.

**Exercise 2.** Let \( X, Y \) be two discrete random variables, each with values in \( \{0, 1\} \).

a) Show that \( X \perp \perp Y \) if \( \mathbb{P}(\{Y = 1\} | \{X = 0\}) = \mathbb{P}(\{Y = 1\} | \{X = 1\}) \).

Let moreover \( Z = X \oplus Y = \begin{cases} 1, & \text{if } X = 1, Y = 0 \text{ or } X = 0, Y = 1, \\ 0, & \text{otherwise.} \end{cases} \)

b) Show that \( X \perp \perp Z \) if \( \mathbb{P}(\{Y = 1\} | \{X = 0\}) = \mathbb{P}(\{Y = 0\} | \{X = 1\}) \).

c) Which assumption guarantees that both \( X \perp \perp Y \) and \( X \perp \perp Z \)?

d) Assume that none of the 3 random variables \( X, Y, Z \) is constant (i.e., takes a single value with probability 1). Can it be that the collection of the three random variables \((X, Y, Z)\) is independent? Justify your answer.

**Exercise 3.** Let \( X_1, X_2 \) be two independent and identically distributed (i.i.d.) \( \mathcal{N}(0, 1) \) random variables. Compute the pdf of \( X_1 + X_2 \) (using convolution).

**Exercise 4.** Let \( X_1, X_2 \) be two i.i.d. random variables such that \( \mathbb{P}(\{X_i = +1\}) = \mathbb{P}(\{X_i = -1\}) = 1/2 \) for \( i = 1, 2 \). Let also \( Y = X_1 + X_2 \) and \( Z = X_1 - X_2 \).

a) Are \( Y \) and \( Z \) independent?

b) Same question with \( X_1, X_2 \) i.i.d. \( \mathcal{N}(0, 1) \) random variables (use here the change of variable formula in order to compute the joint distribution of \( Y \) and \( Z \)).
**Exercise 5.** Let \( \Omega = \mathbb{R}^2 \) and \( \mathcal{F} = \mathcal{B}(\mathbb{R}^2) \). Let also \( X_1(\omega) = \omega_1 \) and \( X_2(\omega) = \omega_2 \) for \( \omega = (\omega_1, \omega_2) \in \Omega \) and let finally \( \mu \) be a probability distribution on \( \mathbb{R} \). We consider below two different probability measures defined on \( (\Omega, \mathcal{F}) \), defined on the “rectangles” \( B_1 \times B_2 \) (Caratheodory’s extension theorem then guarantees that these probability measures can be extended uniquely to \( \mathcal{B}(\mathbb{R}^2) \)).

a) \( \mathbb{P}(1)(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2) \)

b) \( \mathbb{P}(2)(B_1 \times B_2) = \mu(B_1 \cap B_2) \)

In each case, describe what is the relation between the random variables \( X_1 \) and \( X_2 \).