Homework 1

Exercise 1. Let $\Omega = \{1, \ldots, 6\}$ and $A = \{\{1, 2, 3\}, \{1, 3, 5\}\}$.

a) Describe $\mathcal{F} = \sigma(A)$, the $\sigma$-field generated by $A$.

*Hint:* For a finite set $\Omega$, the number of elements of a $\sigma$-field on $\Omega$ is always a power of 2.

b) Give the list of non-empty elements $G$ of $\mathcal{F}$ such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$ 

These elements are called the *atoms* of the $\sigma$-field $\mathcal{F}$ (cf. course). Equivalently, an event $G \in \mathcal{F}$ is *not* an atom if there exists $F \in \mathcal{F}$ such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$.

The atoms of a $\mathcal{F}$ form a *partition* of the set $\Omega$ and they also generate the $\sigma$-field $\mathcal{F}$ in this case. (note also that if $m$ is the number of atoms of $\mathcal{F}$, then the number of elements of $\mathcal{F}$ equals $2^m$)

c) Let $X_1(\omega) = 1_{\{1,2,3\}}(\omega)$, $X_2 = 1_{\{1,3,5\}}(\omega)$ and $Y(\omega) = X_1(\omega) + X_2(\omega)$. Does it hold that $\sigma(Y) = \sigma(X_1, X_2)$?

Exercise 2. Let $\Omega = \{1, \ldots, n\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of subsets of $\Omega$ with $m = O(\log(n))$. Design an algorithm that outputs the list of atoms of the $\sigma$-field $\sigma(\mathcal{A})$. What is the worst-case time-complexity of your algorithm?

Exercise 3. Let now $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}([0, 1])$ be the Borel $\sigma$-field on $[0, 1]$.

a) What are the atoms of $\mathcal{F}$?

b) Is it true in this case that the $\sigma$-field $\mathcal{F}$ is generated by its atoms?

c) Describe the $\sigma$-field $\sigma(\{x\}, x \in [0, 1])$.

Exercise 4. Let $\Omega = \{(i,j) : i,j \in \{1, \ldots, 6\}\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and define the random variables $X_1(i,j) = i$ and $X_2(i,j) = j$.

a) What are $\sigma(X_1)$, $\sigma(X_2)$?

b) Is $X_1 + X_2$ measurable with respect to one of these two $\sigma$-fields?

Exercise 5. Let $\mathcal{F}$ be a $\sigma$-field on a set $\Omega$ and $X_1, X_2$ be two $\mathcal{F}$-measurable random variables taking a finite number of values in $\mathbb{R}$. Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subset \sigma(X_1, X_2)$, i.e., that $X_1, X_2$ carry together at least as much information as $Y$, but that the reciprocal statement is not necessarily true.

a) Provide a non-trivial example of random variables $X_1$, $X_2$ such that $\sigma(Y) = \sigma(X_1, X_2)$.

b) Provide a non-trivial example of random variables $X_1$, $X_2$ such that $\sigma(Y) \neq \sigma(X_1, X_2)$.

c) Assume that there exists $\omega_1 \neq \omega_2$ and $a \neq b$ such that $X_1(\omega_1) = X_2(\omega_2) = a$ and $X_1(\omega_2) = X_2(\omega_1) = b$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?
Exercise 6. Let $\Omega = ]-1, 1]$ and $(X_i, i = 1, \ldots, 4)$ be a family of random variables on $\Omega$ defined as

$$X_i(\omega) = \begin{cases} 
1 & \text{if } \frac{i-1}{4} < \omega \leq \frac{i}{4}, \\
(-1)^i & \text{if } -\frac{i}{4} < \omega \leq -\frac{i-1}{4}, \\
0 & \text{otherwise.}
\end{cases}$$

Describe the $\sigma$-field $F = \sigma(X_i, i = 1, \ldots, 4)$ using its atoms.