Exercise 1: Hopfield network with probabilistic update

So far we have studied Hopfield networks with deterministic activity dynamics. That is, for the same input potential \( h \) a neuron always takes the same state:

\[
S_i(t+1) = \text{sign}(h_i(t)) \quad (1)
\]

In this exercise we model stochastic neurons by replacing that equation with a probabilistic state update:

\[
P\{S_i(t+1) = 1 | h_i(t)\} = g(h_i(t)) \quad (2)
\]

Let’s say we have stored \( M \) patterns \( p^\mu \) in a network of \( N \) neurons. We then set the network to an initial state \( S(t_0) \) that has significant overlap with the third pattern and no overlap with other patterns: \( m^{\mu \neq 3}(t_0) = 0 \). For the deterministic update (eq. 1) we know (either from the textbook or from the proof done last week) we would retrieve pattern \( p^3 \) in a single update: \( m^3(t_0 + 1) = g(m^3(t_0)) = 1 \).

We now study how that result changes in the presence of noisy neurons (eq. 2). Look at figure 1 to get an intuition about the stochastic update.

**Figure 1:** For the analysis of the overlap \( m^3(t + 1) \) it helps to rearrange pattern \( p \) and state \( S \) such that we can identify four sub-populations in the last row. We first split the neurons \( S_i(t) \) into those that \textit{should} be active and those that \textit{should not} be active. All neurons in the same sub-population share the same probabilistic activity dynamics. In the last row, we see four groups of neurons which we label \( \{p_i/S_i(t + 1)\} \): \{on/on\}, \{on/off\}, \{off/on\}, \{off/off\}.

1.1 Derive the overlap \( m^3(t_0 + 1) \) (eq. 3) under the state dynamics of eq. 2. Assume that there’s only overlap with pattern \( p^3 \), and that for each pixel of the pattern 3, the probability to be on is \( P[p^3_i = 1] = 0.5 \)
\[ m^3(t_0 + 1) = g(m^3(t_0)) - g(-m^3(t_0)) \]  

(3)

**Hints:**

1. Use a result we derived earlier: \( h_i(t_0) = p_i^3 m^3(t_0) \).
2. For each of the four groups (see figure 1) find the probabilities for \( P(S_i(t+1)|h_i(t_0)) \).
3. Recall the definition of the overlap \( m^3 \):
   \[ m^3(t_0 + 1) = \frac{1}{N} \sum_{i=1}^{N} p_i^3 S_i(t_0 + 1) \]
4. For large \( N \) we can use the expected number of neurons in each of the four sub populations to express (the expected) overlap \( m^3(t_0 + 1) \).

1.2

(a) In equation 2, what properties should the transfer function \( g \) have?

(b) Use \( g(h) = \frac{1}{2}(\tanh(h) + 1) \) in equation 3. Simplify it, plot the function graph and discuss it.

**Exercise 2: Hopfield, asynchronous update and the energy picture**

Consider a Hopfield network of \( N \) neurons with an **asynchronous** update regime. That is, only one randomly selected neuron \( k \) is updated at each step according to equation 4:

\[
\begin{align*}
S_k(t+1) &= g(h_k(t)) = \text{sign}\left( \sum_{j}^{N} w_{kj} S_j(t) \right) & \text{for exactly one randomly chosen neuron } k \\
S_i(t+1) &= S_i(t) & \text{for all other neurons, } i \neq k
\end{align*}
\]  

(4)

For each state \( S \) of a Hopfield network, we can compute a scalar value, known as the **energy** \( E \) of the network:

\[ E := -\sum_{i}^{N} \sum_{j}^{N} w_{ij} S_i S_j. \]  

(5)

The evolution of the network state and the change of energy are related in an interesting way:

When a network is updated asynchronously then the energy function \( E(S(t)) \) does either decrease or stays at a (local) minimum.

We will now prove this property:

In the trivial case of \( S_k(t+1) = S_k(t) \ \forall k \) the network has reached a stable state and therefore the energy function is stable too: \( \Delta E = E(t+1) - E(t) = 0 \).

Now consider the case of one neuron \( k \) changing its state and proof, in steps 4.1 to 4.3, that the energy decreases:

2.1 The energy \( E(t) \) in eq. 5 is summed over all pre- and post- synaptic neurons \( i \) and \( j \). Rewrite that sum such that the contribution of neuron \( k \) to the total energy \( E \) appears explicitly.

**Hint:** To simplify the resulting expression, remember that in a Hopfield network, the weight are symmetric: \( w_{ij} = w_{ji} \) and there are no self recurrent connections: \( w_{kk} = 0 \)

2.2 Write the change in energy \( \Delta E = E(t+1) - E(t) \) when exactly one neuron \( k \) does changes its state.

2.3 Proof that \( \Delta E < 0 \) when exactly one neuron \( k \) does changes its state under the dynamics of eq. 4.
Exercise 3: Binary codes and spikes

A Hopfield model is specified by a binary variable $S_i \in \{-1, +1\}$, the weights (eq. 6) and the update dynamics (eq. 7).

\[ w_{ij} = c \sum_{\mu=1}^{M} p_i^\mu p_j^\mu \quad \text{with} \quad c = \frac{1}{N} \quad (6) \]

\[ S_i(t+1) = \text{sign} \left( \sum_{j=1}^{N} w_{ij} S_j(t) \right) \quad (7) \]

For an interpretation in terms of spikes it is, however, more appealing to work with a binary variable $\sigma_i$ which is zero or 1.

3.1 Rewrite the Hopfield model in terms of $\sigma_i \in \{0, 1\}$, $S_i = 2\sigma_i - 1$.

3.2 Assume that the patterns have the property $\sum_{i=1}^{N} p_i^\mu = 0 \quad \forall \mu$. Discuss that condition and use it to simplify the update dynamics found in the previous question.

3.3 Assume low-activity patterns $w_{ij} = \sum_{\mu} (\xi_i^\mu - b)(\xi_j^\mu - a)$, where $\xi_i^\mu \in \{0, 1\}$. Can you restrict the weights to excitation only and move negative interaction into a group of inhibitory neurons?